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PLANE AND SPHERICAL TRIGONOMETRY

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A TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY.

BY

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PREFACE.

THIS *Treatise on Trigonometry* has been written for use primarily in the classes of the University of Kansas.

Throughout its entire preparation constant reference has been made to the works of Serret, Lonchampt, Young, Airy, Hind, Beasely, Todhunter, Newcomb, Chauvenet, Olney, "Oliver, Wait, and Jones," Wheeler, Peirce, Loomis, and others. The source of the material is to some extent to be found in those authors. The matter and the methods of presentation are designed to enable the student to become thoroughly acquainted with the principles and applications of Trigonometry; and care has been taken to render the demonstrations of the fundamental propositions as clear and as concise as possible, without in the least affecting their logical accuracy.

In this volume the theory of the science is based upon the analytic method, and every practical formula is illustrated by examples of numerical computation. The *Sets of Examples* given are believed to be sufficient for all practical purposes, furnishing abundance, as well as variety, of work. Answers are not given, in order that the student may test his work by other methods and formulæ.

Special acknowledgments are due Mr. H. B. Newson, Assistant in Mathematics, for a careful review of the manuscript, and for suggestions made.

E. MILLER,
University of Kansas.

DECEMBER, 1891.

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TRIGONOMETRY.

CHAPTER I.

INTRODUCTION.

1. *Trigonometry* is a branch of Mathematics, and comprises all investigations relating to the numerical computation of angles and triangles.

2. *Plane Trigonometry* treats of the solution of plane triangles. It also includes the investigation of all the relations of angles, constituting the *Angular Analysis*, or *Analytical Trigonometry*.

3. *Measurement of Angles.* The *unit* of angular measurement is an angle of one degree.

A *degree is one-ninetieth* of a right angle, or of a quadrant of a circle.

Fractional parts of a degree are represented by *minutes* and *seconds;* thus,

one minute $= \frac{1}{60}$ of a degree;

one second $= \frac{1}{60}$ of a minute.

4. Symbols are used to designate degrees, minutes, and seconds. An angle or arc of 30 degrees, 35 minutes, 50 seconds, is written $30° 35' 50''$. To compute angles or arcs by degrees, minutes, and seconds, is called the *sexagesimal* method, and is in common use.

5. The *centesimal* method consists in dividing a right angle into 100 equal parts, called *grades;* a grade into 100 equal parts, called *minutes;* and a minute into 100 equal parts, called *seconds.*

6. The ordinary conception of an angle, that it must be less than two right angles, is sufficient for geometric purposes and the solution of plane triangles and other rectilinear figures; but Trigonom-

etry shows that angular magnitudes admit of indefinite increase or diminution. That is to say, *an angle or arc may have a value anywhere between* 0 and $+\infty$, or 0 and $-\infty$, in the same manner as *linear*, or *any other kind* of extension.

7. In Art. 3, it is stated that the unit of angular measurement is an angle of one degree. For the solution of triangles, this method is to be preferred. But for certain theoretical purposes, another standard unit may be taken; namely, *the angle at the centre of a circle whose measuring arc is equal in length to the radius.*

If m represents the circumference of a circle, and r the radius, then, by Geometry,

$$m = 2\pi r; \quad \text{from which, if } r = 1,$$
$$m = 2\pi,$$
$$\tfrac{1}{2}\, m = \pi.$$

Now, let $ABCEG$ (Fig. 1) be a circle whose centre is at O. Then, the circumference may be represented by 2π; the semicircumference, by π; and the quadrant ABC, by $\frac{1}{2}\pi$. Let A be a fixed point upon the circumference; P, a point starting at A and moving in the direction of the upper arrow. The arcs AB, ABC, $ABCD$, etc., are, by common consent, *positive arcs.* The arc AB is the measure of the angle AOB; the arc ABC is the measure of the angle AOC; the arc $ABCD$ is the measure of the angle AOD, etc. If the point P, however, should move in the direction of the lower arrow, then the arcs AH, AHG, $AHGF$, etc., are *negative arcs.*

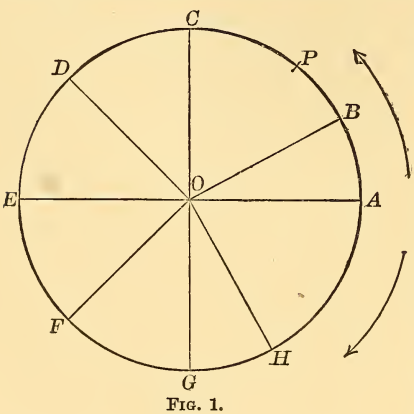

Fig. 1.

8. The diameter AE (Fig. 1) is called the *initial diameter*, and the point A, the *origin of arcs.*

The diameter CG, perpendicular to AE, is the *secondary diameter*, and the point C, the *secondary origin.*

9. The arc AP has no value at the origin; at C, it is $\frac{1}{2}\pi$; at E, π; at G, $\frac{3}{2}\pi$; and at A, or once round, it is 2π. For n circum-

ferences, the arc has a value of $2n\pi$. We may imagine the movement of the point P to continue indefinitely, so that it shall describe an arc composed of many circumferences.

10. The assumption of Art. 7 furnishes an *invariable unit of angular measurement.*

Let AB be the arc of a circle equal in length to the radius CA, the centre being at C.

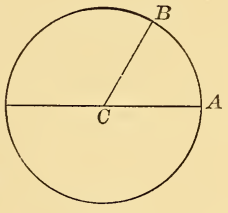

FIG. a'.

Since angles at the centre of a circle are proportional to their subtending arcs, we have

$$\frac{\text{angle } ACB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{circumference}} = \frac{\text{radius}}{2\pi \times \text{radius}} = \frac{1}{2\pi};$$

$$\therefore \text{ angle } ACB = \frac{4 \text{ right angles}}{2\pi} = \frac{2 \text{ right angles}}{\pi}$$

$$= \frac{180°}{\pi} = \frac{180°}{3.14159} = 57° 17' 44.8''$$

= the unit of angular measurement, whatever may be the length of the radius of the circle.

11. *Complementary angles or arcs.* Two angles or arcs, both positive, or one positive and the other negative, are *complementary* when their sum equals a quadrant or $\frac{1}{2}\pi$. The complementary angle of 66° is 24°; of 125° is −35°; of −150° is 240°.

12. *Supplementary angles or arcs.* Two angles or arcs are *supplementary* when their sum equals a semicircumference or π. The supplementary angle of 40° is 140°; of 150° is 30°; of 270° is −90°, and of −190° is 370°.

13. EXAMPLES.

SET I.

1. Construct the angles, 75°; 100°; 120°; 200°; −75°; −100°; −300°; 720°; 1080°; −500°; −630°; 1900°; and name the quadrant to which each belongs.

2. How many degrees in $\frac{1}{4}\pi$? $\frac{3}{4}\pi$? 3π? $3\frac{1}{2}\pi$? -5π? $-\frac{3}{2}\pi$?

3. Express in terms of π, the arcs 15°; 18°; 36°; 45°; 90°; 120°; 150°; 75° 15′ 15″; 270°; 360°; 1800°; −1440°; −85° 25′ 19″.

4. How many times is the *unit arc* contained in 90°? 360°? 300°? $\frac{3}{4}\pi$? 5π? $4n\pi$?

5. What is the complement of 48° 12′? 125° 15′ 16″? $-80°$? $-120°$? π? $-\pi$? $\frac{3}{5}\pi$? $\pi+30°$? $\pi-30°$?

6. What is the supplement of 275° 18′? 48° 12′? $-50°$? $-180°$? π? $-\pi$? $\frac{3}{5}\pi$? 24π? $\pi+50°$? $\frac{3}{2}\pi-50°$? $2n\pi$?

7. The radius of a circle being 10 inches, what is the length of an arc of 5° 30′? 45°? 360°? π? $\frac{3}{4}\pi$? $\pi+$the unit arc? 10 quadrants?

8. The radii of two concentric circles are 5 feet and 10 feet respectively; what is the difference in feet between an arc of 60° on the one and of 60° on the other?

9. The radius of a circle being 10 inches, how many degrees in an arc of 6 inches? Of 29 inches? Of 31.416 inches?

10. If radius equals one foot, how many degrees, minutes, and seconds in an arc of .56 of a foot? Of .275 of a foot? Of .9 of a foot?

11. The radius of a circle being 5 feet, what is the difference in degrees, minutes, and seconds between two arcs, one of which is 15 feet long and the other 12 feet?

12. How many degrees, minutes, and seconds in an arc of 5π? Of 6π? Of $8\frac{1}{2}\pi$? Of 9π?

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

14. The terms used to designate the Trigonometric Functions or Ratios are the words *sine, cosine, tangent, cotangent, secant, cosecant, versed-sine,* and *coversed-sine,* which are written *sin, cos, tan, cot, sec, cosec, vers,* and *covers.*

The investigation of the *properties and the relations* of these functions constitutes the chief part of Trigonometry.

The magnitude of an angle is independent of the lengths of the lines by which it is formed; and accordingly, since the ratios of the sides of a triangle remain unaltered, the magnitude of an angle may be determined by means of the Trigonometric Ratios, of which there are six.

A right triangle being used for this purpose, the definitions of the functions are

1. The *sine* of an angle is the ratio of the *opposite side* to the *hypothenuse.*

2. The *cosine* of an angle is the ratio of the *adjacent side* to the *hypothenuse.*

3. The *tangent* of an angle is the ratio of the *opposite side* to the *adjacent side.*

4. The *cotangent* of an angle is the ratio of the *adjacent side* to the *opposite side.*

5. The *secant* of an angle is the ratio of the *hypothenuse* to the *adjacent side.*

6. The *cosecant* of an angle is the ratio of the *hypothenuse* to the *opposite side.*

The definitions of *versed-sine* and *coversed-sine* are

7. The *versed-sine* equals the difference between *unity* and the *cosine.*

8. The *coversed-sine* equals the difference between *unity* and the *sine.*

15. Let ABC (Fig. 2) be any right triangle, right-angled at C, whose sides are a, b, and c, respectively.

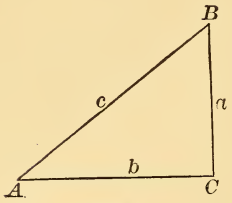

FIG. 2.

Then

$$\left.\begin{aligned}
&(1) \quad \sin A = \frac{a}{c},\\
&(2) \quad \cos A = \frac{b}{c},\\
&(3) \quad \tan A = \frac{a}{b},\\
&(4) \quad \cot A = \frac{b}{a},\\
&(5) \quad \sec A = \frac{c}{b},\\
&(6) \quad \operatorname{cosec} A = \frac{c}{a},\\
&(7) \quad \operatorname{vers} A^* = 1 - \cos A,\\
&(8) \quad \operatorname{covers} A^* = 1 - \sin A.
\end{aligned}\right\} \quad [1]$$

16. Certain other relations are obtained from the foregoing formulæ, as follows:

$$\left.\begin{aligned}
&(1) \quad \tan A \times \cot A = \frac{a}{b} \times \frac{b}{a} = 1; \quad \therefore \tan A = \frac{1}{\cot A},\\
&(2) \quad \sin A \times \operatorname{cosec} A = \frac{a}{c} \times \frac{c}{a} = 1; \quad \therefore \sin A = \frac{1}{\operatorname{cosec} A},\\
&(3) \quad \cos A \times \sec A = \frac{b}{c} \times \frac{c}{b} = 1; \quad \therefore \cos A = \frac{1}{\sec A},\\
&(4) \quad \tan A = \frac{a}{b} = \frac{a}{c} \div \frac{b}{c} \cdot \cdot \cdot \cdot \cdot \cdot \cdot = \frac{\sin A}{\cos A},\\
&(5) \quad \cot A = \frac{b}{a} = \frac{b}{c} \div \frac{a}{c} \cdot \cdot \cdot \cdot \cdot \cdot \cdot = \frac{\cos A}{\sin A}.
\end{aligned}\right\} \quad [2]$$

17.

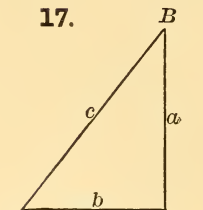

FIG. b'.

In Fig. b', we have

$$\sin A = \frac{a}{c}, \quad \sin B = \frac{b}{c}, \quad \tan A = \frac{a}{b}, \quad \tan B = \frac{b}{a},$$

$$\cos A = \frac{b}{c}, \quad \cos B = \frac{a}{c}, \quad \cot A = \frac{b}{a}, \quad \cot B = \frac{a}{b},$$

$$\sec A = \frac{c}{b}, \quad \sec B = \frac{c}{a}, \quad \operatorname{cosec} A = \frac{c}{a}, \quad \operatorname{cosec} B = \frac{c}{b}.$$

* The *vers* and *covers* being easily found from the *cos* and *sin*, we shall not use them hereafter.

Therefore, since $A + B = 90°$, we find

$$\left.\begin{aligned}
&(1) \quad \sin A = \frac{a}{c} = \cos B, \\
&(2) \quad \cos A = \frac{b}{c} = \sin B, \\
&(3) \quad \tan A = \frac{a}{b} = \cot B, \\
&(4) \quad \cot A = \frac{b}{a} = \tan B, \\
&(5) \quad \sec A = \frac{c}{b} = \operatorname{cosec} B, \\
&(6) \quad \operatorname{cosec} A = \frac{c}{a} = \sec B.
\end{aligned}\right\} \quad [3]$$

18. Since in any right triangle, as ABC (Fig. b'),

$$\overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2,$$

then

$$\frac{\overline{BC}^2}{\overline{AB}^2} + \frac{\overline{AC}^2}{\overline{AB}^2} = 1.$$

Since

$$\frac{BC}{AB} = \sin A, \quad \text{and} \quad \frac{AC}{AB} = \cos A, \quad \text{by [1]},$$

$$\therefore \sin^2 A + \cos^2 A = 1.$$

19. Using the equation $\overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2$, derived from Fig. b', we obtain

$$\frac{\overline{AB}^2}{\overline{AC}^2} = 1 + \frac{\overline{BC}^2}{\overline{AC}^2};$$

$$\therefore \sec^2 A = 1 + \tan^2 A.$$

20. Again, from the equation $\overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2$, we obtain

$$\frac{\overline{AB}^2}{\overline{BC}^2} = 1 + \frac{\overline{AC}^2}{\overline{BC}^2};$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

21. Collecting the results of Arts. 18, 19, 20, and others that follow easily from the same, we shall obtain

$$\left.\begin{aligned}
&(1)\quad \sin^2 A + \cos^2 A = 1,\\
&(2)\quad 1 + \tan^2 A = \sec^2 A,\\
&(3)\quad 1 + \cot^2 A = \operatorname{cosec}^2 A,\\
&(4)\quad \cos A = \sqrt{1 - \sin^2 A},\\
&(5)\quad \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}},\\
&(6)\quad \cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A},\\
&(7)\quad \sec A = \frac{1}{\sqrt{1 - \sin^2 A}},\\
&(8)\quad \operatorname{cosec} A = \frac{1}{\sin A}.
\end{aligned}\right\} \qquad [4]$$

22. The relations already established for angles not exceeding a right angle, hold universally, whatever be the magnitude of an angle, and whether positive or negative.

It must be observed from (1) of [4] that $\sin A = \pm\sqrt{1 - \cos^2 A}$, or $\cos A = \pm\sqrt{1 - \sin^2 A}$ from (4) of [4], has a double sign. In such cases the angle or arc is either positive or negative, and it is necessary to determine from given conditions in any particular case *which sign must be used.*

23. The functional values of the angles 30°, 45°, and 60°, are so often used, an application of the formulæ already obtained will now be made to find those values.

Let ABD (Fig. 3) be an equilateral triangle, divided into two right triangles by BC, a perpendicular from the vertex B upon the base AD.

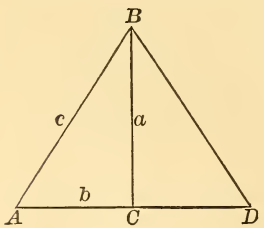

FIG. 3.

In the triangle ABC,

$$AC = \tfrac{1}{2}AD = \tfrac{1}{2}AB$$
$$= \quad b \quad = \tfrac{1}{2}c.$$

The angle $A = 60°$,

and the angle $ABC = 30°$.

By [1], $\cos A = \frac{AC}{AB} = \frac{\frac{1}{2}c}{c} = \frac{1}{2}$. $\therefore$ by [3], $\cos 60° = \frac{1}{2} = \sin 30°$.

By [4], $\sin A = \sqrt{1-\cos^2 A} = \frac{1}{2}\sqrt{3}$. $\therefore$ by [3], $\sin 60° = \frac{1}{2}\sqrt{3} = \cos 30°$.

By [2], $\tan A = \frac{\sin A}{\cos A}$. $\therefore$ by [3], $\tan 60° = \sqrt{3} = \cot 30°$.

By [2], $\cot A = \frac{\cos A}{\sin A}$. $\therefore$ by [3], $\cot 60° = \frac{1}{\sqrt{3}} = \tan 30°$.

By [2], $\sec A = \frac{1}{\cos A}$. $\therefore$ by [3], $\sec 60° = 2 = \operatorname{cosec} 30°$.

By [2], $\operatorname{cosec} A = \frac{1}{\sin A}$. $\therefore$ by [3], $\operatorname{cosec} 60° = \frac{2}{\sqrt{3}} = \sec 30°$.

24. Let the right triangle ABC (Fig. 4) have the angles A and B, each, equal to 45°. Then the sides AC and BC will be equal; i.e. $a = b$.

By Geometry, $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$, or

$b^2 + a^2 = c^2$, or

$2a^2 = c^2$, or

$\frac{a^2}{c^2} = \frac{1}{2}$, or

$\frac{a}{c} = \frac{1}{2}\sqrt{2}$.

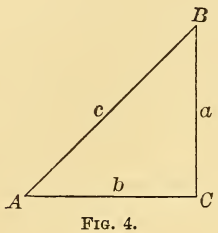

FIG. 4.

$\sin A = \frac{a}{c}$. $\therefore$ by [3], $\sin 45° = \frac{1}{2}\sqrt{2} = \cos 45°$.

$\tan A = \frac{a}{b}$. $\therefore$ by [3], $\tan 45° = 1 = \cot 45°$.

$\sec A = \frac{1}{\cos A}$. $\therefore$ by [3], $\sec 45° = \sqrt{2} = \operatorname{cosec} 45°$.

25. We shall now investigate the Trigonometric Functions of all the quadrants of a circle.

Draw through O, the centre of the circle $ABCD$ (Fig. 5), the diameters AC and BD, at right angles to each other.

Let us take A as the origin of arcs, and those arcs that have the direction of $AMBH \cdots$, etc., as positive arcs, and those whose direction is that of $AM'DT \cdots$, etc., as negative (Art. 7).

We shall further assume that all lines drawn in the *direction* of, or *parallel* to, OA and OB, are *positive*, and those that are drawn in the *direction* of, or *parallel* to, OC and OD, are *negative*.

The arc AMB is the first quadrant; BHC, the second; CTD, the third; and $DM'A$, the fourth.

Designate by θ the positive or negative quantity which represents the variable arc whose extremity M may take upon the circumference all possible positions. Let fall the perpendiculars, MN on OA, and ML on OB. Prolong the radius OM until it meets AE in E. The line OL or its equal MN, with its proper sign, is the *sine* of the arc AM, or θ. The line AE is the tangent of the arc θ, and the line OE is the secant.

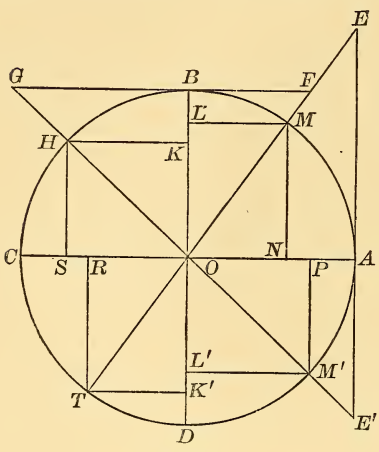

FIG. 5.

MB, the complementary arc of AM, or θ, has for its *sine* the line ON or its equal ML; for its *tangent* BF, and its *secant* OF. We therefore name the *sine*, *tangent*, and *secant*, of the arc MB, the *cosine*, *cotangent*, and *cosecant*, of the arc AM, or θ. Now designating the arc AM, as before, by θ; the arc ABH by θ'; the arc $ABCT$ by θ''; and the arc $ABCDM'$ by θ''', we shall have the following:

	Arc θ.	Arc θ'.	Arc θ''.	Arc θ'''.
sin . . .	$+MN$	$+HS$	$-TR$	$-M'P$
cos . . .	$+ON$	$-OS$	$-OR$	$+OP$
tan . . .	$+AE$	$-AE'$	$+AE$	$-AE'$
cot . . .	$+BF$	$-BG$	$+BF$	$-BG$
sec . . .	$+OE$	$-OE'$	$-OE$	$+OE'$
cosec . . .	$+OF$	$+OG$	$-OF$	$-OG$

26. Therefore, *the sine of an arc is a positive or negative quantity which measures the perpendicular let fall from the extremity of an arc upon the diameter which passes through the origin.*

The tangent of an arc is the positive or negative quantity which measures that portion of the tangent line drawn from the origin of the arc and terminated by the diameter which passes through the extremity of the arc.

The secant of an arc is the positive or negative quantity which measures that portion of the diameter prolonged that is comprised between the centre and the tangent of the arc.

The cosine, cotangent, and cosecant may be described in the same way.

27. We shall now examine in what manner the six trigonometric functions of an arc vary, when that arc varies from 0 to $+\infty$, and from 0 to $-\infty$. If θ increases from 0 to $+\frac{1}{2}\pi$, the six functions remain positive. $\operatorname{Sin}\theta$ increases from 0 to $+1$, while passing through all intermediate values. $\operatorname{Tan}\theta$ increases from 0 to $+\infty$, and $\sec\theta$ from 1 to $+\infty$.

The cosine, cotangent, and cosecant, on the contrary, decrease; that is to say, the $\cos\theta$ decreases from 1 to 0; $\cot\theta$ decreases from $+\infty$ to 0; and $\operatorname{cosec}\theta$ decreases from ∞ to $+1$.

If the arc increases from $\frac{1}{2}\pi$ to π, then $\sin\theta$ decreases from $+1$ to 0; $\tan\theta$ increases from $-\infty$ to 0; $\sec\theta$ increases from $-\infty$ to -1; $\cos\theta$ decreases from 0 to -1; $\cot\theta$ decreases from 0 to $-\infty$; and $\operatorname{cosec}\theta$ increases from $+1$ to $+\infty$.

If the arc increases from π to $\frac{3}{2}\pi$, then $\sin\theta$ decreases from 0 to -1; $\tan\theta$ increases from 0 to $+\infty$; $\sec\theta$ decreases from -1 to $-\infty$; $\cos\theta$ increases from -1 to 0; $\cot\theta$ decreases from $+\infty$ to 0; and $\operatorname{cosec}\theta$ increases from $-\infty$ to -1.

If the arc increases from $\frac{3}{2}\pi$ to 2π, then $\sin\theta$ increases from -1 to 0; $\tan\theta$ increases from $-\infty$ to 0; $\sec\theta$ decreases from $+\infty$ to $+1$; $\cos\theta$ increases from 0 to $+1$; $\cot\theta$ decreases from 0 to $-\infty$; and $\operatorname{cosec}\theta$ decreases from -1 to $-\infty$.

28. If the arc increases from 2π to 4π, or from 4π to 6π, or $\cdots$ to $2n\pi$, the six functions will have periodically the same values and in the same order.

If to the arc any number of circumferences be added, we shall have, whatever be the value of θ, and denoting by n any entire positive or entire negative quantity,

$$\sin(2n\pi+\theta)=\sin\theta, \qquad \cos(2n\pi+\theta)=\cos\theta,$$
$$\tan(2n\pi+\theta)=\tan\theta, \qquad \cot(2n\pi+\theta)=\cot\theta,$$
$$\sec(2n\pi+\theta)=\sec\theta, \qquad \operatorname{cosec}(2n\pi+\theta)=\operatorname{cosec}\theta.$$

29. If θ increases from 0 to $-\infty$, then whatever be the value of θ,

$$\sin(-\theta)=-\sin\theta, \qquad \cos(-\theta)=\cos\theta,$$
$$\tan(-\theta)=-\tan\theta, \qquad \cot(-\theta)=-\cot\theta,$$
$$\sec(-\theta)=\sec\theta, \qquad \operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta.$$

30. If θ be any arc, then θ and $\pi+\theta$ terminate at the extremities of the same diameter, and we shall have,

$$\sin(\pi+\theta)=-\sin\theta, \qquad \cos(\pi+\theta)=-\cos\theta,$$
$$\tan(\pi+\theta)=\tan\theta, \qquad \cot(\pi+\theta)=\cot\theta,$$
$$\sec(\pi+\theta)=-\sec\theta, \qquad \operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta.$$

31. If in the equations of Art. 30 we change θ to $-\theta$, we shall have,

$$\sin(\pi-\theta)=\sin\theta, \qquad \cos(\pi-\theta)=-\cos\theta,$$
$$\tan(\pi-\theta)=-\tan\theta, \qquad \cot(\pi-\theta)=-\cot\theta,$$
$$\sec(\pi-\theta)=-\sec\theta, \qquad \operatorname{cosec}(\pi-\theta)=\operatorname{cosec}\theta.$$

From which it follows that *if two arcs are supplementary, their sines and their cosecants are equal and of the same sign, but their cosines, tangents, cotangents, and secants are equal and of contrary signs.*

From the two preceding groups of equations, denoting by n any entire quantity positive or negative, we shall have,

$$\sin[(2n+1)\pi\pm\theta]=\mp\sin\theta, \qquad \cos[(2n+1)\pi\pm\theta]=-\cos\theta,$$
$$\tan(n\pi\pm\theta)=\pm\tan\theta, \qquad \cot(n\pi\pm\theta)=\pm\cot\theta,$$
$$\sec[(2n+1)\pi\pm\theta]=-\sec\theta, \qquad \operatorname{cosec}[(2n+1)\pi\pm\theta]=\mp\operatorname{cosec}\theta.$$

32. It is very important to remark that each of the trigonometric functions of an arc θ takes all values of which it is susceptible in the indefinite variation of θ throughout an interval of two quadrants.

33. To the functions $x=\sin\theta$, $x=\tan\theta$, $x=\cos\theta$, $x=\sec\theta$, $x=\cot\theta$, etc., corresponds another class of functions, which are usually written by the Germans and the French, $\theta=\text{arc-sin}\,x$,

$\theta = \text{arc-tan}\, x$, $\theta = \text{arc-cos}\, x$, $\theta = \text{arc-sec}\, x$, $\theta = \text{arc-cot}\, x$, etc., and by others $\theta = \sin^{-1} x$, $\theta = \tan^{-1} x$, $\theta = \cos^{-1} x$, $\theta = \sec^{-1} x$, $\theta = \cot^{-1} x$, etc. The German and French method is to be preferred.

It is easily seen that $\theta = \text{arc-sin}\, x$, $\theta = \text{arc-tan}\, x$, etc., are not entirely determined, for they admit of an indefinite number of values for each value of x. The expressions $\text{arc-sin}\, x$, $\text{arc-tan}\, x$, $\text{arc-cot}\, x$, and $\text{arc-cosec}\, x$, become completely determined if their values are constantly comprised between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; and in like manner, $\text{arc-cos}\, x$ and $\text{arc-sec}\, x$ will be determined if their values are constantly comprised between 0 and π. With these restrictions, the expressions $\text{arc-sin}\, x$, $\text{arc-tan}\, x$, $\text{arc-cos}\, x$, $\text{arc-cot}\, x$, $\text{arc-sec}\, x$, and $\text{arc-cosec}\, x$, may be considered as functions of x.

34. EXAMPLES.

SET II.

Note. — Throughout this book, unless otherwise stated, radius is taken equal to *unity*.

1. Construct:

$\sin\theta = \frac{2}{3}$,	$\sin\theta = \pi$,	$\theta = \text{arc-sin}\,\frac{1}{2}$,
$\cos\theta = \frac{2}{3}$,	$\cos\theta = \pi$,	$\theta = \text{arc-tan}\,\frac{3}{4}$,
$\tan\theta = 1$,	$\sin\theta = -\frac{1}{8}$,	$\theta = \text{arc-cos}\,\frac{2}{\sqrt{3}}$,
$\sec\theta = 5$,	$\tan\theta = -\frac{9}{10}$,	$\theta = \text{arc-sec}\,2$,
$\tan\theta = \infty$,	$\cot\theta = 6$,	$\theta = \text{arc-cot}\,3$.

2. Determine the values of the Trigonometrical Ratios for an angle of 585°. Also for an angle of 690°. Also for an angle of 930°. Also for an angle of 6420°.

3. Find all the angles between 0 and 900° which satisfy $\tan\theta = 1$. Find all the angles between 0 and 900° which satisfy $\cos^2\theta = \frac{1}{2}$.

4. Find the values of the other functions of θ, when

$\sin\theta = \frac{2}{3}$,	$\sin 135° = \frac{1}{2}\sqrt{2}$,
$\sec\theta = 4$,	$\cos 120° = -\frac{1}{2}$,
$\cot\theta = \frac{m}{n}$,	$\tan 1440° = 0$,
$\tan\theta = -\frac{1}{3}$,	$\cot 540° = \infty$,
$\tan\theta = -3$,	$\sin(2\pi + 30°) = \frac{1}{2}$,
$\cos\theta = -\frac{1}{3}$,	$\cos(\pi + 90°) = 0$,
$\csc\theta = -1$,	$\sec(2\pi + 90°) = \infty$.

5. Given $\tan\theta = \sin\theta$; find the angle θ.

6. Given $\cos\theta = \tan\theta$; find the angle θ.

7. Given $\operatorname{cosec} 45° = \sec(180° - \theta)$; find the angle θ.

8. Given $\cot(m+n)\theta = \tan\theta$; find θ.

9. Given $\sin\frac{1}{2}\theta = \cos(90 - 5\frac{1}{2}\theta)$; find θ.

10. Given $10\sin\theta = 2\tan\theta$; find $\cos\theta$, $\sin\theta$, and $\tan\theta$.

11. Given $2\sin^2\theta = 3\cos\theta$; find θ.

12. Given $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$; find θ.

13. Given $\tan\theta + \cot\theta = 2$; find θ.

14. Show that $\tan\theta + \cot\theta = \dfrac{\sec^2\theta + \operatorname{cosec}^2\theta}{\sec\theta \operatorname{cosec}\theta}$.

15. Find the values of θ that will satisfy $\sin\theta\cos\theta = \tan\theta$.

16. Show that $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$.

17. When is $\sin\theta + \cos\theta = 1$? When equal to -1? When is $\sin^2\theta = 1$? $\cos^2\theta = 1$? When is $\tan^2\theta + \cot^2\theta = 2$?

18. Find the value of $\sin\dfrac{3\pi}{12}$. Of $\sin(-4\pi)$. Of $\cos\dfrac{\pi}{4}$. Of $\cos(-6\pi)$.

19. Construct arc-tan 1; arc-cos $(-\frac{1}{5})$; arc-tan (-5); arc-cosec (-5).

20. Trace the changes in the sign and value of $\cos\theta - \sin\theta$, as θ passes from 0 to 2π. Also of $\tan\theta + \cot\theta$.

21. Find all the angles between 0 and $\dfrac{10\pi}{3}$ which satisfy the relation $\cos^2\theta = \frac{1}{2}$.

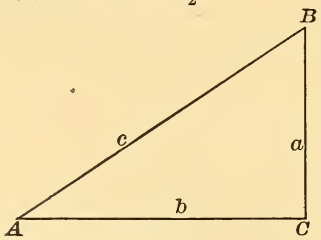

FIG. 6.

22. Find $\sin\theta$ and $\cos\theta$, when $a + b = \frac{5}{4}c$. In this example use Fig. 6, in which the angle A is denoted by θ, and the angle C is a right angle.

23. Find the value of θ, when $\tan 6\theta = 1$.

24. Find $\sin\theta$, when $\sec\theta + \tan\theta = 2$.

25. Show that $1 + \sin\theta = \dfrac{\cos^2\theta}{1 - \sin\theta}$.

26. Letting $n = 0, 1, 2, 3, 4, 5, 6$, in succession, show that

$$\sin 4n\frac{\pi}{2} = 0; \quad \sin(4n+2)\frac{\pi}{2} = 0; \quad \sin(4n+3)\frac{\pi}{2} = -1;$$

$$\cos(4n+2)\frac{\pi}{2} = -1; \quad \text{and} \quad \tan(2n+1)\frac{\pi}{2} = \infty.$$

27. Show that $\sin(270° - \theta) = -\cos\theta$.

28. Show that $\cos(270° - \theta) = -\sin\theta$.

29. Show that $\cos(270° + \theta) = \sin\theta$.

30. Show that $\sin(360° - \theta) = -\sin\theta$.

31. Show that $\cos(360° - \theta) = \cos\theta$.

32. Show that $\dfrac{\sec\theta + \operatorname{cosec}\theta}{\sec\theta - \operatorname{cosec}\theta} = \dfrac{1 + \cot\theta}{1 - \cot\theta} = \dfrac{\tan\theta + 1}{\tan\theta - 1}$.

33. From the table of natural functions, find the

sin of 15°,	cos of 25° 15′ 18″,
sin of 18° 15′,	cos of 135° 25′ 20″,
sin of 75° 10′ 35″,	tan of 60° 55′ 43″.

34. From the table of logarithmic functions, find the logarithmic sin, cos, tan, and cot of

18° 18′ 18″,	100° 50′,
50° 0′ 20″,	175° 14′ 25″,
150° 15′ 25″,	75° 16′ 40″.

35. From the table of natural functions, find the angle whose sin is .25256; cos, .78543; tan, 3.14156; and cot, .56789.

36. From the table of logarithmic functions, find the angle whose logarithmic sin is 9.12345; cos, 9.34567; tan, 10.43216; and cot, 10.23456.

CHAPTER III.

TRIANGLES AND POLYGONS.

35. To solve a triangle is to calculate the numerical value of its unknown elements, when a sufficient number of parts are given.

In every triangle there are three angles and three sides.

In a right triangle, the right angle is always known; and to solve such a triangle, either *an oblique angle and a side* or *two sides* must be given.

When the given parts are the three angles of a triangle, the problem is indeterminate; that is to say, an indefinite number of triangles may, each, satisfy the conditions.

36. CASE I.

Given two sides of a right triangle; to find the other parts.

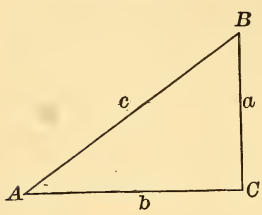

FIG. 7.

Let ABC (Fig. 7) be a triangle, right-angled at C. Let a and b be the given sides; then, by Art. 15, we have

$$\tan BAC = \frac{a}{b}, \quad \text{or,} \quad \cot ABC = \frac{a}{b}.$$

$$\therefore \ ^*\log\tan BAC = \log a - \log b + 10,$$

$$\text{or} \quad \log\cot ABC = \log a - \log b + 10.$$

By Art 15, $\sin BAC = \frac{a}{c}$; $\therefore \log c = \log a - \log\sin BAC + 10.$

Checks. $BAC + ABC = 90°$, and $c = \sqrt{a^2 + b^2}$.

EXAMPLE. Given $a = 15.5$, and $b = 18.2$. Find the side c and the angles A and B.

Solution by logarithms:

$$(1) \quad \log a = 1.190332$$
$$\log b = \underline{1.260071}$$
$$\log\tan A = \overline{9.930261} = \log\cot B.$$
$$\therefore \ A = 40° 25' 9''.6, \quad \text{and} \quad B = 49° 34' 50''.4.$$

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* See Chapter XIII.

$$(2)\quad \log a = 1.190332$$
$$\log \sin A = \underline{9.811828}$$
$$\log c = 1.378504.$$
$$\therefore\ c = 23.905.$$

Checks. $40^\circ\,25'\,9''.6 + 49^\circ\,34'\,50''.4 = 90^\circ$; and

$$23.905 = \sqrt{\overline{15.5}^2 + \overline{18.2}^2}.$$

37. CASE II.

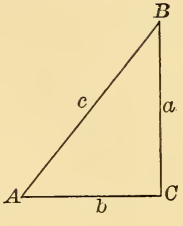
Given the hypothenuse and one side; to find the other parts.

Let c and a be given.

We have, to determine the unknown parts,

$$\sin A = \cos B = \frac{a}{c},$$

and $$b^2 = c^2 - a^2 = (c+a)(c-a).$$



The direct determination of the angle B is obtained by the formula for the cosine; but if the hypothenuse differs little from the given side, as is sometimes the case, then the angle B is not *exactly* determined. To avoid the difficulty, we may first obtain the side b, and then, by Art. 15, (3), [1], calculate the angle B from $\tan B = \frac{b}{a}$.

By logarithms:

$$(1)\quad \log b = \tfrac{1}{2}[\log(c+a) + \log(c-a)],$$
$$(2)\quad \log \tan B = \log b - \log a + 10,$$
$$(3)\quad \log \tan A = \log a - \log b + 10.$$

Checks. $A + B = 90^\circ$, and $c = \sqrt{a^2 + b^2}$.

EXAMPLE. Given $c = 5892.51$, and $a = 5439.24$. Find b, A, and B.

Solution: $c + a = 11331.75$, $c - a = 453.27$.

By (1), $\log(c+a) = 4.054297$
$\log(c-a) = 2.656357$
$2)\overline{6.710654}$
$\log b = 3.355327.$
$\therefore\ b = 2266.35.$

By (2), $\log b = 3.355327$
$\log a = \underline{3.735538}$
$\log \tan B = 9.619789$
$\therefore\ B = 22^\circ\,37'\,11''.4.$

$$\text{By (3)},\quad \log a = \ 3.735538$$
$$\log b = \ 3.355327$$
$$\log\tan A = 10.380211.$$
$$\therefore\ A = 67^\circ\,22'\,48''.6.$$

Checks. $67^\circ\,22'\,48''.6 + 22^\circ\,37'\,11''.4 = 90^\circ$; and

$$5892.51 = \sqrt{5439.24^2 + 2266.35^2}.$$

38. CASE III.

Given an acute angle and one side; to find the other parts.

Let the angle A and the side a be given. Then the required parts may be determined by the formulæ

$$(1)\ A + B = 90^\circ;\quad (2)\ \sin A = \frac{a}{c};\quad (3)\ \tan A = \frac{a}{b}.$$

(2) and (3) changed to the form of logarithms become

$$\log c = \log a - \log\sin A + 10,$$
$$\log b = \log a - \log\tan A + 10.$$

39. CASE IV.

Given an acute angle and the hypothenuse; to find the other parts.

Let the angle A and the hypothenuse c be given. Then the required parts may be determined by the formulæ

$$(1)\ A + B = 90^\circ;\quad (2)\ \sin A = \frac{a}{c};\quad (3)\ \cos A = \frac{b}{c}.$$

(2) changed to the logarithmic form becomes

$$\log a = \log\sin A + \log c - 10;$$

(3) changed to the logarithmic form becomes

$$\log b = \log\cos A + \log c - 10.$$

40. Whatever may be the given parts of a right triangle, the required parts can always be found by the use of the formulæ

$$(1)\ c = \sqrt{a^2 + b^2};\quad (2)\ A + B = 90^\circ;\quad (3)\ [1] \text{ and } [3].$$

41. *Isosceles Triangles.*

A perpendicular let fall from the vertex of an isosceles triangle upon the base divides the triangle into two equal right triangles. *The triangle is solved by solving the right triangles.*

42. *Oblique Triangles.*

A perpendicular let fall from the vertex of an oblique triangle upon the side opposite divides the triangle into two right triangles. *The oblique triangle is solved by solving the two right triangles.*

43. *Regular Polygons.*

Let *ABCDEF* (Fig. 8) be a regular polygon.

Lines drawn from the centre *O* to the middle points of the sides, as at *G*, are radii of the inscribed circle; and lines drawn from the centre to the vertices of the polygon are radii of the circumscribed circle. In this manner a regular polygon is divided into twice as many right triangles as the polygon has sides. Let n be the number of sides of a regular polygon; then the angle AOG will equal $\frac{360°}{2n} = \frac{180°}{n}$, and the angle OAG will equal $90° - \frac{180°}{n}$.

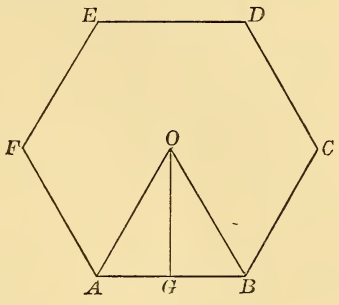


FIG. 8.

If one of the two radii, *OA* or *OG*, or the side *AB*, be given, the angles having been determined as above, the remaining parts of the polygon may be found by the formulæ for right triangles.

44. *Area of Right Triangles.*

The base b and the perpendicular a of a right triangle being given, the area, by Geometry, is equal to $\frac{1}{2}ab$.

45. EXAMPLES.

SET III.

1. Given $A = 50°$, $c = 15$; to find B, a, b.
2. Given $A = 80°$, $b = 40$; to find B, a, c.
3. Given $A = 70° 15'$, $a = 225$; to find B, b, c.
4. Given $A = 35° 35' 35''$, $B = 54° 24' 25''$; to find a, b, c.
5. Given $A = 59° 37' 42''$, $c = 1785.395$; to find B, a, b.
6. Given $B = 40° 45' 43''$, $a = 15.15$; to find A, b, c.
7. Given $B = 50° 50' 50''$, $b = 201.356$; to find A, a, c.

8. Given $B = 75° 30' 38''$, $c = 400$; to find A, a, b.

9. Given $a = 65$, $b = 72$; to find A, B, c.

10. Given $a = 2269$, $c = 3269$; to find A, B, b.

11. Given $a = .00075$, $A = 75°$; to find B, b, c.

12. Given $b = 99.5$, $c = 100$; to find A, B, a.

13. The hypothenuse of a right triangle is 18 feet long, and one of the acute angles is 4 times the other. Find the angles and sides.

14. In a right triangle whose hypothenuse is 15, and the angle $A = \text{arc-tan}\, 3$; what are the other parts?

15. In a right triangle the side $a = 515.5$; and the angle B, $50° 45'$. Find the other parts.

16. The uniform grade of a railroad track 500 feet long is 4°. What is the elevation at the end?

17. What are the values of the trigonometric functions of a right triangle whose sides are 30, 40, 50, respectively? Are there any other right triangles that give the same values?

18. The hypothenuse of a right triangle is 6 times the base. Find the angles.

19. A tower 300 feet high casts a shadow 150 feet long upon the horizontal plane upon which it stands. Find the altitude of the sun.

Isosceles Triangles.

20. One of the two equal sides of an isosceles triangle is 18 feet long; one of the equal angles, 30°. Find the *third angle*, the *base*, and the *perpendicular* from the third angle upon the base.

21. Given the third angle of an isosceles triangle, and the perpendicular from that angle upon the base. Find the *equal angles*, the *equal sides*, and the *base*.

22. A chord 30 feet long is drawn in a circle whose diameter is 60 feet. Find the angle at the centre.

23. A circle whose diameter is 100 feet has two chords, one of which subtends an angle at the centre of 30°, and the other an angle of 50°. Find the difference in length between the two chords.

Regular Polygons.

24. Find the side of an equilateral triangle inscribed in a circle whose radius is 10.

25. Find the side of a regular decagon inscribed in a circle whose radius is 10.

26. Find the radius of a circle in which is inscribed a regular pentagon whose perimeter is 50 feet.

27. Find the radius of a circle in which is inscribed a regular dodecagon whose perimeter is 120 feet.

28. Find the side of a regular pentadecagon inscribed in a circle whose radius is 50.

Areas of Right Triangles and Regular Polygons.

29. Find the area of a right triangle, the angle A being 75°, and the side a equal to 50.

30. Find the area of a right triangle, A being equal to 50° 50′, and c equal to 100.

31. The area of a right triangle is 500, and the hypothenuse 200. Find the angles and sides.

32. Find the area of a regular pentagon inscribed in a circle whose radius is 10 feet.

33. Find the area of the space between a regular decagon inscribed in, and a regular decagon circumscribed about, a circle whose radius is 10 feet.

34. How much space between the circumference of a circle of 20 feet diameter, and the perimeter of a regular dodecagon inscribed in the circle?

Miscellaneous.

35. Solve the right triangle, when $A = 30°$, $b = 100$, and $a = 40$.

36. $A = 18°$, and $c = 4 + \sqrt{80}$. Find a, b, and B.

37. Show whether there can be a right triangle when

$$\log a + 10 = \log b + \log \sin A.$$

38. If $c \cos B = b \cos C$, show that the triangle is isosceles; that is, that $b = c$.

39. The elevation of a tower is 30° to a man 6 feet high at a distance of 140 feet from the foot of the tower. Find the height of the tower.

40. A man's shadow is twice his height. What is the altitude of the sun?

41. The sides of a right triangle are 20 and 32. Show that $A = \text{arc-tan}\,\frac{8}{5}$, $B = \text{arc-tan}\,\frac{5}{8}$, and the hypothenuse $= 4\sqrt{89}$.

42. Show that if $2 \sin\theta = \tan\theta$, $\theta = 0$, or 60°.

43. If $6\cot^2\theta - 4\cos^2\theta = 1$, then $\theta = \pm 60°$.

44. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\theta = 90°$.

45. Show that $\sin\left(\frac{4n+3}{2}180° + \theta\right) = -\cos\theta$.

46. The difference of the lengths of the shadows of a vertical stick is 10 feet, when the sun's altitude is 45° and 30° respectively. Find the length of the stick.

47. The difference of two acute angles of a right-angled triangle is $\theta°$. Find the angles.

48. Show that $\frac{\sin 45° - \sin 30°}{\sin 45° + \sin 30°} = (\sec 45° - \tan 45°)^2$.

49. The length of a kite string is 300 yards, and the elevation of the kite is 32°. Find its height.

50. If a rectangle, 3 feet long by 2 feet broad, be taken as the unit of surface, what quantity will represent two square feet?

CHAPTER IV.

FORMULÆ FOR THE SUM OR DIFFERENCE OF TWO ANGLES OR ARCS.

46. *To determine the sine and the cosine of the sum of two angles or arcs, the sines and the cosines of the angles or arcs being known.*

Let AB and BC be two positive arcs whose sum does not exceed $\frac{1}{2}\pi$. The origin of arcs (Fig. 9) being at A, represent the arc AB by θ, and BC by θ'. The arc BC may be considered as having its origin at B and its extremity at C.

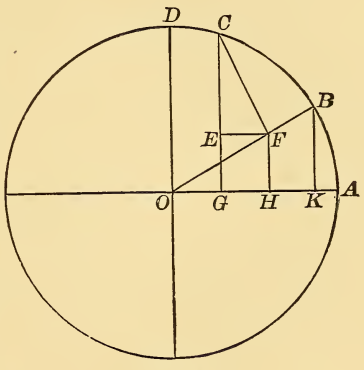


FIG. 9.

Then $\theta + \theta' =$ the arc AC.

Draw CF perpendicular to the radius OB, and CG, FH, BK, perpendicular to OA, and FE parallel to OA.

Then
$$\sin\theta = BK,$$
$$\cos\theta = OK,$$
$$\sin\theta' = CF,$$
$$\cos\theta' = OF.$$

(1) $\sin(\theta+\theta') = CG = CE + FH,$

(2) $\cos(\theta+\theta') = OG = OH - EF.$

Radius is, in all these demonstrations, taken equal to unity. The similar triangles OBK and OFH give

$$\frac{FH}{BK} = \frac{OH}{OK} = \frac{OF}{OB}.$$

From the first and third ratios we obtain

(3) $$FH = \frac{BK \cdot OF}{OB} = \sin\theta\cos\theta'.$$

From the second and third ratios we obtain

$$(4)\quad OH = \frac{OK \cdot OF}{OB} = \cos\theta \cos\theta'.$$

From the similar triangles OBK and CEF we have

$$\frac{CE}{OK} = \frac{FE}{BK} = \frac{CF}{OB}.$$

$$\therefore\ (5)\quad CE = \frac{OK \cdot CF}{OB} = \cos\theta \sin\theta'.$$

$$(6)\quad FE = \frac{BK \cdot CF}{OB} = \sin\theta \sin\theta'.$$

Now, substitute (3) and (5) in (1), and (4) and (6) in (2); then

$$\sin(\theta + \theta') = \sin\theta \cos\theta' + \cos\theta \sin\theta', \qquad [5]$$

$$\cos(\theta + \theta') = \cos\theta \cos\theta' - \sin\theta \sin\theta'. \qquad [6]$$

47. *To determine the sine and the cosine of the difference of two angles or arcs, the sines and cosines of the angles or arcs being known.*

If in [5] and [6], θ' be changed to $-\theta'$, then

$$\sin(\theta - \theta') = \sin\theta \cos\theta' - \cos\theta \sin\theta', \qquad [7]$$

$$\cos(\theta - \theta') = \cos\theta \cos\theta' + \sin\theta \sin\theta'. \qquad [8]$$

48. *To find the tangent and cotangent of the sum of two angles or arcs.*

By Art. 16, (4), [2],

$$\tan(\theta + \theta') = \frac{\sin(\theta + \theta')}{\cos(\theta + \theta')}.$$

By [5] and [6],

$$\frac{\sin(\theta + \theta')}{\cos(\theta + \theta')} = \frac{\sin\theta \cos\theta' + \cos\theta \sin\theta'}{\cos\theta \cos\theta' - \sin\theta \sin\theta'}.$$

Dividing both terms of the right member by $\cos\theta \cos\theta'$, then

$$\tan(\theta + \theta') = \frac{\tan\theta + \tan\theta'}{1 - \tan\theta \tan\theta'}. \qquad [9]$$

Again, by Art. 16, (5), [2],

$$\cot(\theta+\theta')=\frac{\cos(\theta+\theta')}{\sin(\theta+\theta')}.$$

By [6] and [5],

$$\frac{\cos(\theta+\theta')}{\sin(\theta+\theta')}=\frac{\cos\theta\cos\theta'-\sin\theta\sin\theta'}{\sin\theta\cos\theta'+\cos\theta\sin\theta'}.$$

Dividing both terms of the right member by $\sin\theta\sin\theta'$, then

$$\cot(\theta+\theta')=\frac{\cot\theta\cot\theta'-1}{\cot\theta'+\cot\theta}. \qquad [10]$$

49. *To find the tangent and cotangent of the difference of two angles or arcs.*

If in [9] and [10], θ' be changed to $-\theta'$, then

$$\tan(\theta-\theta')=\frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'}. \qquad [11]$$

$$\cot(\theta-\theta')=\frac{1+\cot\theta\cot\theta'}{\cot\theta'-\cot\theta}. \qquad [12]$$

50. Formulæ [5], [7], [9], and [11] have been demonstrated on the supposition that $\theta+\theta'$ is not greater than $\frac{1}{2}\pi$. It can be easily shown that they are generally true; that is to say, that the *sines, cosines, tangents,* and *cotangents* of the sum of two angles or arcs, greater than $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, 2π, or $2n\pi$, n being any finite integer, are all represented by the four formulæ named.

For example,

$$\sin(\tfrac{3}{2}\pi+\theta+\theta')=\sin(\tfrac{3}{2}\pi+\theta)\cos\theta'+\cos(\tfrac{3}{2}\pi+\theta)\sin\theta'.$$

51. It is easily shown that

$$\sin(\theta+\theta'+\theta'')=\sin\theta\cos\theta'\cos\theta''+\sin\theta'\cos\theta\cos\theta''+\sin\theta''\cos\theta\cos\theta'-\sin\theta\sin\theta'\sin\theta''. \qquad [13]$$

$$\cos(\theta+\theta'+\theta'')=\cos\theta\cos\theta'\cos\theta''-\cos\theta\sin\theta'\sin\theta''-\cos\theta'\sin\theta\sin\theta''-\cos\theta''\sin\theta\sin\theta'. \qquad [14]$$

Knowing thus the formulæ for the sine and cosine of the sum of three angles or arcs, we may obtain those which will give the sine and the cosine of the sum of four angles or arcs, or of five, or six, or of n angles or arcs.

52. *Important formulæ deduced from* [5], [6], [7], and [8].

$$\left.\begin{aligned} &\text{From [5] and [6],} && \sin(\theta+\theta')+\sin(\theta-\theta') = 2\sin\theta\cos\theta', \\ & && \sin(\theta+\theta')-\sin(\theta-\theta') = 2\cos\theta\sin\theta'. \\ &\text{From [7] and [8],} && \cos(\theta+\theta')+\cos(\theta-\theta') = 2\cos\theta\cos\theta', \\ & && \cos(\theta+\theta')-\cos(\theta-\theta') = -2\sin\theta\sin\theta'. \end{aligned}\right\} \quad [15]$$

Now, let $\theta+\theta'=\phi$, and $\theta-\theta'=\phi'$; then we shall have

$$\theta=\tfrac{1}{2}(\phi+\phi'), \text{ and } \theta'=\tfrac{1}{2}(\phi-\phi'),$$

and the preceding formulæ, [15], become

$$\left.\begin{aligned} &(1)\ \sin\phi+\sin\phi' = 2\sin\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi'), \\ &(2)\ \sin\phi-\sin\phi' = 2\sin\tfrac{1}{2}(\phi-\phi')\cos\tfrac{1}{2}(\phi+\phi'), \\ &(3)\ \cos\phi+\cos\phi' = 2\cos\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi'), \\ &(4)\ \cos\phi'-\cos\phi = 2\sin\tfrac{1}{2}(\phi+\phi')\sin\tfrac{1}{2}(\phi-\phi'). \end{aligned}\right\} \quad [16]$$

These last formulæ are frequently employed; they serve to express the sum or difference of two sines or of two cosines in terms of the product of sines, or cosines, or of a sine and a cosine.

53. We may express by a product the sum or the difference of a sine and of a cosine.

We have

$$\cos\phi \pm \sin\phi' = \sin(\tfrac{1}{2}\pi-\phi) \pm \sin\phi';$$

and using [16],

$$\left.\begin{aligned} &\cos\phi+\sin\phi' = 2\sin\left(\tfrac{1}{4}\pi-\frac{\phi-\phi'}{2}\right)\cos\left(\tfrac{1}{4}\pi-\frac{\phi+\phi'}{2}\right), \\ &\cos\phi-\sin\phi' = 2\sin\left(\tfrac{1}{4}\pi-\frac{\phi+\phi'}{2}\right)\cos\left(\tfrac{1}{4}\pi-\frac{\phi-\phi'}{2}\right). \end{aligned}\right\} \quad [17]$$

54. It is easily seen that, by division, a new set of important formulæ may be derived from [16].

$$\left.\begin{aligned}
&(1)\quad \frac{\sin\phi+\sin\phi'}{\sin\phi-\sin\phi'}=\frac{\sin\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi')}{\sin\tfrac{1}{2}(\phi-\phi')\cos\tfrac{1}{2}(\phi+\phi')}=\frac{\tan\tfrac{1}{2}(\phi+\phi')}{\tan\tfrac{1}{2}(\phi-\phi')},\\
&(2)\quad \frac{\sin\phi+\sin\phi'}{\cos\phi+\cos\phi'}=\frac{\sin\tfrac{1}{2}(\phi+\phi')}{\cos\tfrac{1}{2}(\phi+\phi')}=\tan\tfrac{1}{2}(\phi+\phi'),\\
&(3)\quad \frac{\sin\phi+\sin\phi'}{\cos\phi-\cos\phi'}=\frac{\cos\tfrac{1}{2}(\phi-\phi')}{\sin\tfrac{1}{2}(\phi-\phi')}=\cot\tfrac{1}{2}(\phi-\phi'),\\
&(4)\quad \frac{\sin\phi-\sin\phi'}{\cos\phi+\cos\phi'}=\frac{\sin\tfrac{1}{2}(\phi-\phi')}{\cos\tfrac{1}{2}(\phi-\phi')}=\tan\tfrac{1}{2}(\phi-\phi'),\\
&(5)\quad \frac{\sin\phi-\sin\phi'}{\cos\phi'-\cos\phi}=\frac{\cos\tfrac{1}{2}(\phi+\phi')}{\sin\tfrac{1}{2}(\phi+\phi')}=\cot\tfrac{1}{2}(\phi+\phi'),\\
&(6)\quad \frac{\cos\phi+\cos\phi'}{\cos\phi'-\cos\phi}=\frac{\cos\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi')}{\sin\tfrac{1}{2}(\phi+\phi')\sin\tfrac{1}{2}(\phi-\phi')}\\
&\qquad\qquad=\cot\tfrac{1}{2}(\phi+\phi')\cot\tfrac{1}{2}(\phi-\phi').
\end{aligned}\right\}\ [18]$$

55. *Functions of double angles or arcs.*

If in formulæ [5], [6], [9], and [10], θ' be made equal to θ, then,

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad [19]$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad [20]$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}, \qquad [21]$$

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}. \qquad [22]$$

56. *Functions of half angles or arcs.*

By [19], $\quad 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = \sin\theta.$

By Art. 21, (1), [4], $\quad \cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta = 1;$

from which, by addition and subtraction, we obtain

$$(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta)^2 = 1 + \sin\theta,$$

$$(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta)^2 = 1 - \sin\theta.$$

$$\therefore\ \cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta = \pm\sqrt{1+\sin\theta},$$

$$\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta = \pm\sqrt{1-\sin\theta};$$

$$\therefore \sin\tfrac{1}{2}\theta = \pm\tfrac{1}{2}(\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}). \qquad [23]$$

$$\cos\tfrac{1}{2}\theta = \pm\tfrac{1}{2}(\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}). \qquad [24]$$

Resuming formulæ [20], and (1) of [4], Art. 21,

$$\cos^2\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}\theta = \cos\theta,$$

$$\cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta = 1,$$

we obtain by addition and subtraction, and extracting the square root,

$$\cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}, \qquad [25]$$

$$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}. \qquad [26]$$

It will be noticed that [23] and [24] give four values for $\sin\frac{1}{2}\theta$, and four values for $\cos\frac{1}{2}\theta$, respectively; and it is remarkable that the values of $\cos\frac{1}{2}\theta$ are precisely the same as those of $\sin\frac{1}{2}\theta$.

By (4) of [2], Art. 16, and [25] and [26],

$$\tan\tfrac{1}{2}\theta = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}},$$

which becomes, when both terms of the radical are multiplied by

$$\sqrt{1+\cos\theta}, \text{ or by } \sqrt{1-\cos\theta},$$

$$\tan\tfrac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}. \qquad [27]$$

In like manner, $\cot\frac{1}{2}\theta = \dfrac{1+\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1-\cos\theta}$. [28]

57. If we wish to express $\sin 2\theta$ or $\cos 2\theta$ in terms of $\sin\theta$ or $\cos\theta$ only, we must replace $\sin\theta$ by its value $\pm\sqrt{1-\cos^2\theta}$, and $\cos\theta$ by $\pm\sqrt{1-\sin^2\theta}$.

Formulæ [19] will then become

$$\sin 2\theta = \pm 2\sin\theta\sqrt{1-\sin^2\theta} = \pm 2\cos\theta\sqrt{1-\cos^2\theta}, \qquad [29]$$

and [20] will become

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1. \qquad [30]$$

These results are such that if we know the values of $\sin\theta$ and $\cos\theta$, then $\cos 2\theta$ will be completely determined, but the $\sin 2\theta$ will have two equal values, one of which is positive and the other negative. It is important to remember the effect of the double sign, in order to determine the quadrant to which the function belongs.

58. Examples.

SET IV.

1. Show that $\sin(45° + \theta) = \cos(45 - \theta)$.

2. Show that $\sin\theta = \dfrac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$.

3. Show that $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \tan 2\theta + \sec 2\theta$.

4. Show that $\dfrac{\cos(\theta + 45°)}{\cos(\theta - 45°)} = \sec 2\theta - \tan 2\theta$.

5. Find the value of θ in $90° + \text{arc-sin}\,\theta = \text{arc-tan}\,\theta$.

6. Show that $\dfrac{\sin(\theta \pm \theta')}{\sin\theta\sin\theta'} = \cot\theta' \pm \cot\theta$.

7. Show that $\dfrac{\cos(\theta \pm \theta')}{\sin\theta\sin\theta'} = \cot\theta\cot\theta' \mp 1$.

8. Show that $\dfrac{\sin(\theta + \theta')}{\sin(\theta - \theta')} = \dfrac{\tan\theta + \tan\theta'}{\tan\theta - \tan\theta'}$.

9. Show that $\dfrac{\sin(\theta \pm \theta')}{\cos(\theta \mp \theta')} = \dfrac{\tan\theta \pm \tan\theta'}{1 \pm \tan\theta\tan\theta'}$.

10. Show that $\tan(\theta + \theta' + \theta'')$
$$= \frac{\tan\theta + \tan\theta' + \tan\theta'' - \tan\theta\tan\theta'\tan\theta''}{1 - \tan\theta\tan\theta' - \tan\theta\tan\theta'' - \tan\theta'\tan\theta''}.$$

11. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

12. Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

13. Show that $\sin 4\theta = 4(\sin\theta - 2\sin^3\theta)\cos\theta$.

14. Show that $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$.

15. Show that $\tan 5\theta = \dfrac{\tan 4\theta + \tan\theta}{1 - \tan 4\theta\tan\theta} = \dfrac{\tan 3\theta + \tan 2\theta}{1 - \tan 3\theta\tan 2\theta}$.

16. Find the value of θ in $3\tan\theta\tan 3\theta + 1 = 0$, when $\theta < \frac{\pi}{2}$. Also, when $\theta < \pi$.

17. Show that $2\tan 2\theta = \tan(45° + \theta) - \tan(45° - \theta)$.

18. Show that $\dfrac{\cos\theta - \cos 3\theta}{\sin 3\theta - \sin\theta} = \tan 2\theta$.

19. Find the value of θ when $\sin 4\theta + \sin\theta = 0$.

20. Find the value of θ when $\tan\theta + \tan(\frac{1}{2}\pi + \theta) = 2$.

21. Find $\tan\theta$ when $\tan\theta + ab\cot\theta = a + b$.

22. If $\theta + \theta' + \theta'' = 180°$, show that

$$\tan\theta + \tan\theta' + \tan\theta'' = \tan\theta\tan\theta'\tan\theta''.$$

23. Take (10), and show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

24. If $\theta + \theta' + \theta'' = 90°$, show that

$$1 = \tan\theta\tan\theta' + \tan\theta\tan\theta'' + \tan\theta'\tan\theta''.$$

25. If $\theta + \theta' + \theta'' = 180°$, show that

$$\sin\theta + \sin\theta' + \sin\theta'' = 4\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta''.$$

26. Show that $\cos\theta + \cos(120° - \theta) + \cos(120° + \theta) = 0$.

27. Show that $4\sin\theta\sin(60° - \theta)\sin(60° + \theta) = \sin 3\theta$.

28. Show that $\dfrac{\tan\theta \pm \tan\theta'}{\cot\theta \pm \cot\theta'} = \pm\tan\theta\tan\theta'$.

29. Find the value of $\sin(45° + 30°)$.

30. The functions of 30° being given (Art. 23), find those of 15°.

31. Given $\sin\theta = m\sin\theta'$, and $\tan\theta = n\tan\theta'$; find $\sin\theta$ and $\cos\theta'$.

32. If $\theta + \theta' + \theta'' = 180°$, show that

$$\sin^2\theta + \sin^2\theta' + \sin^2\theta'' - 2\cos\theta\cos\theta'\cos\theta'' = 2.$$

33. Given $\sin 210° = -\frac{1}{2}$; find $\cos 105°$.

34. Given $\tan 2\theta = -\frac{24}{7}$; find $\sin\theta$ and $\cos\theta$.

35. Find the value of θ when

$$\sin 3\theta + \sin 2\theta + \sin\theta = 0.$$

CHAPTER V.

ANGLES, OBLIQUE TRIANGLES, AND CIRCLES.

59. *Relations between the angles and the sides of an oblique triangle.*

THEOREM I. *In every plane triangle, the sides are proportional to the sines of the angles opposite.*

Let ABC (Fig. 10) be a triangle whose angles A and C are acute; from the vertices B and C draw the perpendiculars BD to the base AC, and CE to the side AB.

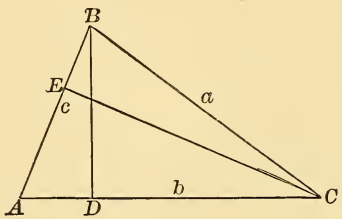


FIG. 10.

The two right triangles ABD and DBC will give, by Art. 15,

$$BD = c \sin A,$$

and $$BD = a \sin C.$$

(1) $$\therefore\ c \sin A = a \sin C.$$

The two right triangles CEA and CEB will, in like manner, give

$$CE = b \sin A, \quad \text{and} \quad CE = a \sin B.$$

(2) $$\therefore\ b \sin A = a \sin B.$$

Hence $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad [31]$$

If an angle of the triangle ABC (Fig. 11), C, for example, be obtuse, the perpendicular BD will fall upon the base produced.

Since two supplementary angles have the same sine, the triangles ABD and CBD give, as before,

$$BD = c \sin A = a \sin C;$$

from which we conclude that the preceding formula is general.

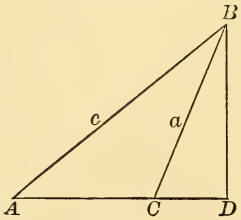


FIG. 11.

We have, therefore, the three following relations between the angles and the sides of a triangle:

$$\left.\begin{aligned} A+B+C&=180°,\\ \frac{a}{\sin A}&=\frac{b}{\sin B},\\ \frac{a}{\sin A}&=\frac{c}{\sin C}. \end{aligned}\right\} \qquad [32]$$

It is readily seen that these three formulæ may be written

$$A=180°-B-C,$$

$$b=\frac{a\sin B}{\sin(B+C)},$$

$$c=\frac{a\sin C}{\sin(B+C)}.$$

60. THEOREM II. *In any plane triangle the square of any side is equal to the sum of the squares of the other two sides, less twice the product of these two sides by the cosine of their included angle.*

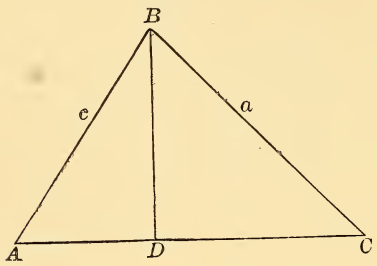


FIG. 12.

Let ABC (Fig. 12) be a triangle whose angle C is acute. Draw from the vertex B, to the base AC, the perpendicular BD. By Geometry, we have

$$c^2=a^2+b^2-2b\times CD;$$

but the right triangle BCD gives, Art 15,

$$CD=a\cos C.$$

$$\left.\begin{aligned} \therefore\ c^2&=a^2+b^2-2ab\cos C.\\ \text{In like manner,}\quad a^2&=b^2+c^2-2bc\cos A,\\ b^2&=a^2+c^2-2ac\cos B. \end{aligned}\right\} \qquad [33]$$

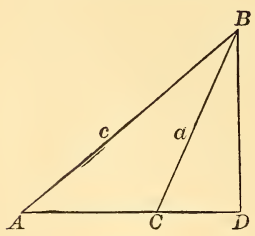


FIG. 13.

If the angle C (Fig. 13) be obtuse, we have

$$c^2=a^2+b^2+2b\times CD.$$

The right triangle CBD gives

$$CD=a\cos BCD.$$

But $\cos BCD = \cos(180° - BCA) = -\cos BCA.$

$\therefore\ c^2 = a^2 + b^2 - 2ab\cos BCA.$

$\therefore$ the formulæ [33] are general.

61. If each of the angles A, B, and C be made successively a right angle, what form will [31] assume? What form will [33] assume?

62. THEOREM III. *In any plane triangle, any side is equal to the sum of the other two, each multiplied by the cosine of the angle which that side makes with the first side.*

Let BD be a perpendicular from the vertex B upon the base AC of the triangle ABC. (Fig. 14).

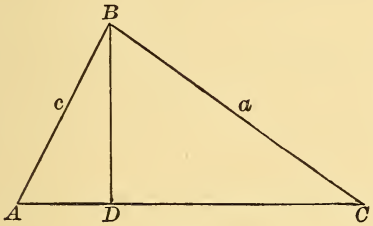


FIG. 14.

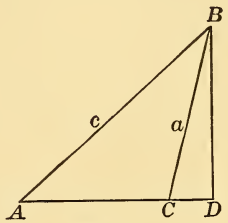


FIG. 15.

We have, if the angles A and C are acute,

$$b = AD + DC;$$

but if one of the angles, as C, is obtuse (Fig. 15), then

$$b = AD - DC.$$

In the first case, $DC = a\cos C.$

In the second case, $DC = b\cos(180° - C) = -b\cos C.$

And in both cases, $AD = c\cos A.$

We shall have

$$\left.\begin{aligned} &(1)\ a = b\cos C + c\cos B,\\ &(2)\ b = c\cos A + a\cos C,\\ &(3)\ c = a\cos B + b\cos A. \end{aligned}\right\} \qquad [34]$$

63. Formulæ [34] may be obtained from [33] by adding those of [33] two and two, and reducing the results. Reciprocally, the first of [33] may be obtained from [34] by adding the three equations of [34], after having multiplied (1) by a, (2) by b, and (3) by $-c$.

In like manner the others may be obtained.

64. THEOREM IV. *In any triangle, the sum of two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

From the fundamental formulæ [31],

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

we obtain

$$(1)\quad \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)}{\sin C},$$

$$(2)\quad \frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)}{\sin C};$$

whence, dividing (1) by (2), we have

$$\frac{a+b}{a-b} = \frac{\tan\tfrac{1}{2}(A+B)}{\tan\tfrac{1}{2}(A-B)}. \qquad [35]$$

65. *Other formulæ relating to oblique triangles.*

The relation $a^2 = b^2 + c^2 - 2bc\cos A,$

when changed to the form

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

will give the angle A; and the other two formulæ of [33] will, in like manner, give the angles B and C.

We shall now proceed to establish formulæ for the functions of $\frac{1}{2}A$, $\frac{1}{2}B$, and $\frac{1}{2}C$.

From [25] and [26] we have

$$(1)\quad \cos\tfrac{1}{2}A = \sqrt{\frac{1+\cos A}{2}},$$

$$(2)\quad \sin\tfrac{1}{2}A = \sqrt{\frac{1-\cos A}{2}};$$

and, substituting for $\cos A$ its value $\frac{b^2+c^2-a^2}{2bc}$ in (1), we have

(1) $\cos\frac{1}{2}A$

$$=\sqrt{\frac{2bc+b^2+c^2-a^2}{4bc}}=\sqrt{\frac{(b+c)^2-a^2}{4bc}}=\sqrt{\frac{(a+b+c)(-a+b+c)}{4bc}};$$

and in (2) also,

(2) $\sin\frac{1}{2}A$

$$=\sqrt{\frac{2bc-b^2-c^2+a^2}{4bc}}=\sqrt{\frac{a^2-(b-c)^2}{4bc}}=\sqrt{\frac{(a+b-c)(a-b+c)}{4bc}}.$$

Now divide (2) by (1);

then (3) $\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}=\tan\frac{1}{2}A=\sqrt{\frac{(a+b-c)(a-b+c)}{(a+b+c)(-a+b+c)}}.$

If we put $a+b+c=2s,$

then $-a+b+c=2(s-a),$

$a-b+c=2(s-b),$

$a+b-c=2(s-c).$

Now substitute these values in equations (2), (1), and (3), and developing similar equations for $\frac{1}{2}B$ and $\frac{1}{2}C$, we shall have three systems of formulæ:

$$\left.\begin{aligned}\sin\tfrac{1}{2}A&=\sqrt{\frac{(s-b)(s-c)}{bc}},\\ \sin\tfrac{1}{2}B&=\sqrt{\frac{(s-a)(s-c)}{ac}},\\ \sin\tfrac{1}{2}C&=\sqrt{\frac{(s-a)(s-b)}{ab}}.\end{aligned}\right\}\qquad[36]$$

$$\left.\begin{aligned}\cos\tfrac{1}{2}A&=\sqrt{\frac{s(s-a)}{bc}},\\ \cos\tfrac{1}{2}B&=\sqrt{\frac{s(s-b)}{ac}},\\ \cos\tfrac{1}{2}C&=\sqrt{\frac{s(s-c)}{ab}}.\end{aligned}\right\}\qquad[37]$$

$$\left.\begin{aligned} \tan\tfrac{1}{2}A &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \\ \tan\tfrac{1}{2}B &= \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \\ \tan\tfrac{1}{2}C &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \end{aligned}\right\} \quad [38]$$

In all of [36], [37], and [38], the radicals must be taken with the sign +; for the half-angles of any triangle are, each, less than 90°, and consequently their trigonometric functions are positive.

66. *Area of Triangles.*

Let the perpendicular BD be drawn from the vertex B of the triangle ABC to the base AC or AC produced.

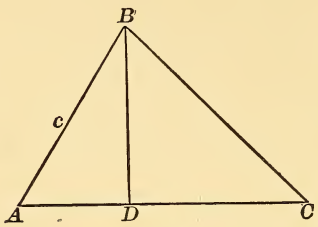


FIG. 16.

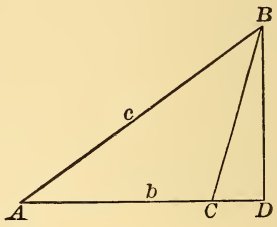


FIG. 17.

Designating the area of a triangle by K, we have

$$K = \tfrac{1}{2}b \times BD.$$

But from the right triangle ABD we have $BD = c \sin A$, in either figure.

$$\therefore\ K = \tfrac{1}{2}bc \sin A. \quad [39]$$

Therefore, *the area of a triangle equals one-half the product of two sides multiplied by the sine of the angle included between those sides.*

Show that *the area of any quadrilateral equals one-half the product of its diagonals multiplied by the sine of the angle between those diagonals.*

67. *To find the area of a triangle in terms of its three sides.*

If in formula [39] we replace c by its value taken from [31],

$$\frac{c}{b}=\frac{\sin C}{\sin B}=\frac{\sin C}{\sin(A+C)}, \text{ whence } c=\frac{b\sin C}{\sin(A+C)},$$

[39] will become $$K=\tfrac{1}{2}\frac{b^2\sin A\sin C}{\sin(A+C)}.$$

Finally, from equations,

$$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-c)(s-b)}{bc}},$$

and

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}},$$

established in Art. 65, we obtain

$$\sin A=2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A=2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc};$$

and if we replace $\sin A$, in formula [39], by this value, then

$$K=\sqrt{s(s-a)(s-b)(s-c)}. \qquad [40]$$

By means of [36], [37], [38], and [40], it may be shown that

$$\left.\begin{aligned}
&(1)\ \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C=\frac{(s-a)(s-b)(s-c)}{abc}=\frac{K^2}{sabc},\\
&(2)\ \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C=\frac{s\sqrt{s(s-a)(s-b)(s-c)}}{abc}=\frac{Ks}{abc},\\
&(3)\ \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C\ \ldots\ldots\ldots=\frac{K}{s^2}.
\end{aligned}\right\}\ [41]$$

Show that

$$(1)\quad \frac{s}{c}=\frac{\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B}{\sin\tfrac{1}{2}C},$$

$$(2)\quad \frac{s-c}{c}=\frac{\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B}{\sin\tfrac{1}{2}C},$$

$$(3)\quad \frac{s-b}{c}=\frac{\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B}{\cos\tfrac{1}{2}C},$$

$$(4)\quad \frac{s-a}{c}=\frac{\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B}{\cos\tfrac{1}{2}C}.$$

68. *Radii of circumscribed, inscribed, and escribed circles.*

(1) Having circumscribed the circle whose centre is O (Fig. 18) about the triangle ABC, draw from the vertex B the diameter $BD = 2R$, and draw CD. The angle $A =$ the angle D; the triangle BCD is a right triangle.

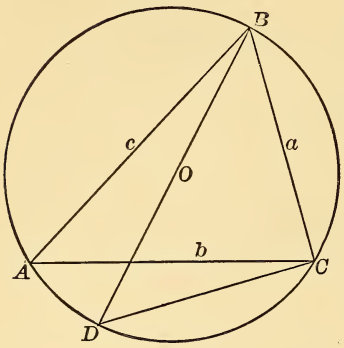


FIG. 18.

$$\therefore\ a = 2R\sin D = 2R\sin A.$$

$$\therefore R = \frac{a}{2\sin A}.$$

Multiplying both terms of $\frac{a}{2\sin A}$ by bc, we have

$$R = \frac{abc}{2bc\sin A};$$

then, multiplying both [39] and [40] by 4, we have

$$4K = 2bc\sin A = 4\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore\ R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \qquad [42]$$

(2) Let the circle (Fig. 19), whose centre is O and radius $= r$, be inscribed in the triangle ABC. The lines OA, OB, and OC divide the triangle into three triangles, whose bases are a, b, and c, respectively, and whose common altitude is r.

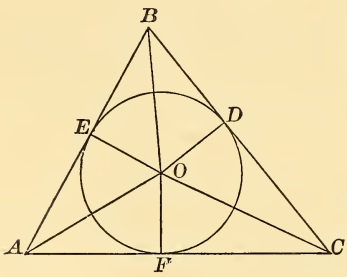


FIG. 19.

As before, let

$K =$ the area of ABC,

and $s = \frac{1}{2}(a+b+c)$;

then $K = rs$,

and $r = \frac{K}{s}$.

But by [40] $K = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore\ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad [43]$$

(3) If we designate by r', r'', and r''' the radii of the escribed circles, that is to say, the circles which touch respectively the sides

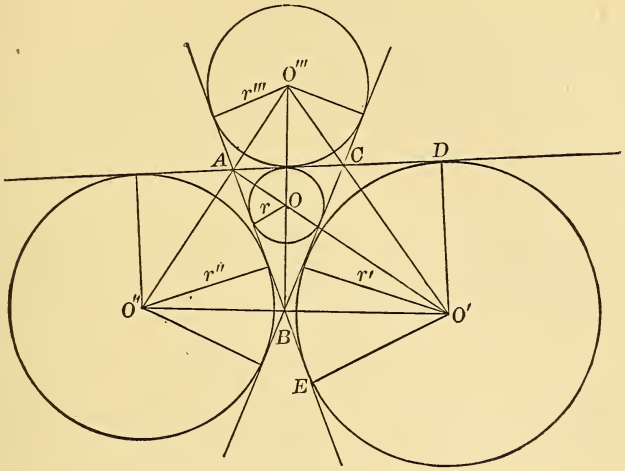


FIG. 20.

a, b, and c, and the prolongations of the other two sides, then it can be easily shown (Fig. 20) that from the area of the triangle ABC, which equals

$$\sqrt{s(s-a)(s-b)(s-c)} = K, \qquad [40], \text{ Art. } 67.$$

$$= (s-a)r' = (s-b)r''' = (s-c)r'',$$

we can obtain

$$\left.\begin{aligned} r' &= \frac{K}{s-a} = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \\ r'' &= \frac{K}{s-c} = \sqrt{\frac{s(s-a)(s-b)}{s-c}}, \\ r''' &= \frac{K}{s-b} = \sqrt{\frac{s(s-a)(s-c)}{s-b}}. \end{aligned}\right\} \qquad [44]$$

The formulæ just obtained may, by Art. 65, be written

$$\left.\begin{aligned} r' &= s\tan\tfrac{1}{2}A, \\ r'' &= s\tan\tfrac{1}{2}C, \\ r''' &= s\tan\tfrac{1}{2}B. \end{aligned}\right\} \qquad [45]$$

From the foregoing formulæ many others may be derived, among which are the following:

$$\left.\begin{aligned} \frac{1}{r} &= \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}, \\ K &= \sqrt{r\,r'\,r''\,r'''}, \\ 4R^* &= r' + r'' + r''' - r. \end{aligned}\right\} \qquad [46]$$

Exercises.

SET V.

1. If a', b', c' be the perpendiculars from the centre of the circumscribed circle upon the sides a, b, c, respectively, of the triangle ABC; then

$$\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'} = \frac{1}{4} \cdot \frac{abc}{a'b'c'}.$$

2. Prove that the product of the perpendiculars of a triangle from the angles on the opposite sides equals

$$\frac{[(a+b+c)r]^3}{abc},$$

r being the radius of the inscribed circle.

3. If R and r be radii of circles circumscribed about and inscribed in a regular polygon whose side is $2a$; then

$$R^2 - r^2 = a^2.$$

4. If r, r' be radii of the circumscribed and inscribed circles of a regular polygon of n sides, show that

$$r + r' = a \cot \frac{\pi}{2n},$$

where $2a$ is one of the sides.

5. If the radius of a circle be r, show that the length of an arc which subtends an angle of $\theta°$ at the centre is $\frac{\pi r \theta}{180}$.

---

* R = the radius of the circumscribed circle.

CHAPTER VI.

SOLUTION OF OBLIQUE TRIANGLES.

69. Case I. *To solve a triangle, when two angles and one side are given.*

The unknown angle is obtained by the formula

$$A + B + C = 180°.$$

If a be the given side, we obtain the sides b and c by the formulæ

$$b = \frac{a \sin B}{\sin A},$$

$$c = \frac{a \sin C}{\sin A}.$$

70. Example. Given $A = 81° 47' 12''.5$,

$B = 38° 12' 47''.5$,

$a = 7012.24$; to find C, b, and c.

(1) $A + B + C = 180°$. $\therefore\ C = 60°$.

(2) $b = \dfrac{a \sin B}{\sin A}$.

$\log a = 3.845856$

$\log \sin B = 9.791402$

$\operatorname{colog} \sin A = 0.004477$

$\log b = 3.641735$

$\therefore\ b = 4382.65$.

(3) $c = \dfrac{a \sin C}{\sin A}$.

$\log a = 3.845856$

$\log \sin C = 9.937530$

$\operatorname{colog} \sin A = 0.004477$

$\log c = 3.787863$

$\therefore\ c = 6135.71$.

71. Case II. *To solve a triangle, when two sides and the angle opposite one of them are given.*

If a, b, and A are the given elements, we may determine the unknown parts by the formulæ

$$\sin B = \frac{b \sin A}{a},$$

$$A + B + C = 180°,$$

$$c = \frac{a \sin C}{\sin A}.$$

When the angle B differs but little from 90°, it cannot be exactly determined by means of the sine.

In the case of a right triangle there can be no uncertainty as to the angles, because one being a right angle, either of the other two angles can be exactly determined from the formula $A + B = 90°$.

But when an angle of an oblique triangle is determined from its sine or cosecant, then uncertainty may exist, since there are two angles each less than 180° having a given sine or cosecant.

There can be no doubt as to the cosine, tangent, cotangent, or secant.

In this case, the solution may determine two triangles, one triangle, or none.

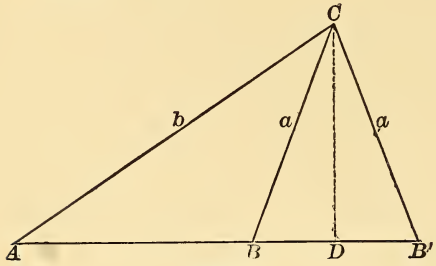


Fig. 21.

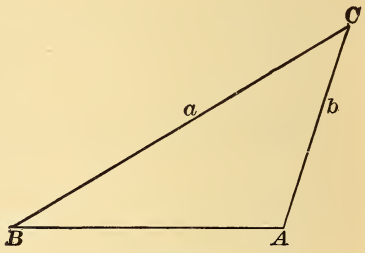


Fig. 22.

Suppose A to be the given angle, and a and b the given sides. The angle B is found by the formula

$$\sin B = \frac{b \sin A}{a}.$$

(1) If $\frac{b \sin A}{a} < 1$ (Fig. 21), then two angles, B and B', will be determined, one $> 90°$, and the other $< 90°$.

Now, if $a > b$, then the angle $A >$ the angle B; therefore B must be an acute angle, and there will be but one triangle.

But if $a < b$, then either B or B' will meet the conditions, and two triangles will be determined.

The angle C will have two values, and the side c also will have two values.

(2) If $\frac{b \sin A}{a} = 1$, it is clear that the figure is then a right triangle.

(3) If $\frac{b \sin A}{a} > 1$, no triangle exists.

(4) If $A > 90°$ (Fig. 22), and $a > b$, there will be but one triangle, as in (1).

But if $A > 90°$ and $a < b$, there will be no triangle.

EXAMPLE. Given $A = 27° 47' 44''.77$,
$a = 2199.12$,
$b = 2513.28$; to find B, C, and c.

$$(1)\quad \sin B = \frac{b \sin A}{a}.$$

$$\begin{aligned} \log b &= 3.400240 \\ \log \sin A &= 9.668685 \\ \operatorname{colog} a &= \bar{4}.657751 \\ \log \sin B &= \bar{1}.726676 \end{aligned}$$

$$\therefore B = 32° 12' 15''.23$$
and $147° 47' 44''.77$.

$$(2)\quad C = 180° - (A + B).$$

$\therefore$ when $B = 32° 12' 15''.23$, $C = 120° 0' 0''$;
and when $B = 147° 47' 44''.77$, $C = 4° 24' 30''.66$.

$$(3)\quad c = \frac{a \sin C}{\sin A}.$$

$\log a = 3.342248$	$\log a = 3.342248$
$\log \sin C = 9.937530$	$\log \sin C = 8.885735$
$\operatorname{colog} \sin A = 0.331314$	$\operatorname{colog} \sin A = 0.331314$
$\log c = 3.611092$	$\log c = 2.559297$
$\therefore c = 4084.08.$	$\therefore c = 362.493.$

72. CASE III. *To solve a triangle, when two sides and their included angle are given.*

Let a, b, and C be the given parts; then, to find the angles A and B, we may employ formula [35],

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)},$$

and the side c by

$$c = \frac{a \sin C}{\sin A}.$$

EXAMPLE. Given $a = 153,$
$b = 137,$
$C = 40° 33' 12''.$

(1) $a + b = 290,$ $a - b = 16,$ and $\frac{1}{2}(A + B) = 69° 43' 24''.$

$$\log(a - b) = 1.204120$$
$$\log \tan \tfrac{1}{2}(A + B) = 10.432446$$
$$\operatorname{colog}(a + b) = \overline{3}.537602$$
$$\log \tan \tfrac{1}{2}(A - B) = 9.174168$$
$$\therefore \tfrac{1}{2}(A - B) = 8° 29' 37'',$$
$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A = 78° 13' 1'',$$
$$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B = 61° 13' 47''.$$

(2) $$c = \frac{a \sin C}{\sin A}.$$

$$\log a = 2.184691$$
$$\log \sin C = 9.813018$$
$$\operatorname{colog} \sin A = 0.009249$$
$$\log c = 2.006958$$
$$\therefore c = 101.616.$$

Problems under this case may be readily solved by the formula

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}, \qquad [33], \text{ Art. } 60,$$

when the side c is required.

The angles A and B may then be found by [31].

73. *To solve a triangle, when the three sides are given.*

The formulæ of [33], which determine the angles A, B, and C, by the sides, are not adapted to logarithmic computation; but those of Art. 65 may be employed; particularly the last set, [38], which determine the angles $\frac{1}{2}A$, $\frac{1}{2}B$, and $\frac{1}{2}C$, by their tangents.

EXAMPLE. Given $a = 701.224,$
$b = 438.265,$
$c = 613.571;$ to find A, B, and C.

(1) Let $s = \frac{1}{2}(a + b + c) = 876.530,$
$s - a = 175.306,$
$s - b = 438.265,$
$s - c = 262.959.$

(2) $\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

$\log(s-b) = 2.641736$
$\log(s-c) = 2.419888$
$\operatorname{colog}(s-a) = \bar{3}.756203$
$\operatorname{colog} s = \bar{3}.057233$
$2)\overline{\bar{1}.875061}$
$\log\tan\frac{1}{2}A = \bar{1}.937530$
$\therefore \frac{1}{2}A = 40^\circ\,53'\,36''.22.$
$A = 81^\circ\,47'\,12''.44.$

(3) $\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$.

$\log(s-a) = 2.243796$
$\log(s-c) = 2.419888$
$\operatorname{colog}(s-b) = \bar{3}.358263$
$\operatorname{colog} s = \bar{3}.057233$
$2)\overline{\bar{1}.079180}$
$\log\tan\frac{1}{2}B = \bar{1}.539590$
$\therefore \frac{1}{2}B = 19^\circ\ 6'\,23''.77.$
$B = 38^\circ\,12'\,47''.54.$

(4) $\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$.

$\log(s-a) = 2.243796$
$\log(s-b) = 2.641736$
$\operatorname{colog}(s-c) = \bar{3}.580111$
$\operatorname{colog} s = \bar{3}.057233$
$2)\overline{\bar{1}.522876}$
$\log\tan\frac{1}{2}C = \bar{1}.761438$
$\therefore \frac{1}{2}C = 30^\circ,$
$C = 60^\circ.$

74. Examples.

SET VI.

1. The sides of a triangle are in the ratio of $2 : \sqrt{6} : 1+\sqrt{3}$. Determine the angles.

2. The sides of a triangle are 32, 40, and 66. Find the greatest angle.

3. Given $b = 14$, $c = 11$, and $A = 60^\circ$; to find B.

4. Given $a = 70$, $b = 35$, and $C = 36^\circ\,52'\,12''$. Solve the triangle.

5. Given $a = 18$, $c = 2$, and $B = 55^\circ$. Find the other angles.

6. $A = 45^\circ\,50'$, $B = 60^\circ\,15'$, and $c = 25$. Solve the triangle.

7. $B = 25^\circ\,18'$, $C = 130^\circ\,15'$, and $b = 480$. Solve the triangle.

8. Two sides of a triangle are 3 feet and 5 feet respectively, and the included angle is 120°. Solve the triangle.

9. The sides of a triangle are 4, 5, and 6. Find the angles.

10. In a triangle ABC, the side AB is 254.3, the side AC 396.8, and the angle B 94° 21′. Solve the triangle.

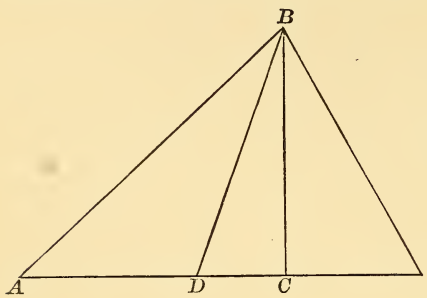
11. Given $a = 201$, $b = 140$, and $B = 36° 44'$; to solve the triangle. Are there two triangles?

12. The sides of a triangle are 18, 19, and 20. Find the angles.

13. Given $A = 32°$, $a = 40$, and $b = 50$; to solve the triangle. Are there two triangles?

14. Given $A = 18° 52' 13''$, $a = 27.465$, and $b = 13.189$. Solve the triangle.

15. The distance $AB = 600$ yards; the angle $BAC = 57° 35'$, and the angle $ABC = 64° 51'$. Find AC and BC.



16. Find BC, the height of a hill above a horizontal plane; AD being equal to 1000 feet; the angle A, 15° 36′; and ABD, 27° 29′.

17. A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane on which it stands. Find the altitude of the sun.

18. A person standing at the edge of a river observes that the top of a tower on the opposite edge makes an angle of 55° with a horizontal line drawn from his eye; walking 30 feet back from the edge, the angle then is 48°. Find the breadth of the river.

19. Find the area of a plane triangle whose sides are 24 feet, 30 feet, and 18 feet, respectively.

20. The sides of a triangle are 3, 5, and 6. Compare the radii of the circumscribed and inscribed circles.

21. The sides of a triangle are 75 and 85, and the difference of the angles opposite those sides is 60°. Find all the angles.

22. Find the area of a triangular field, one of whose sides is 45 rods long, and the two adjacent angles, respectively, 70° and 69° 40′.

23. How far off may a hill 500 feet high be seen, if the radius of the earth be 3962 miles?

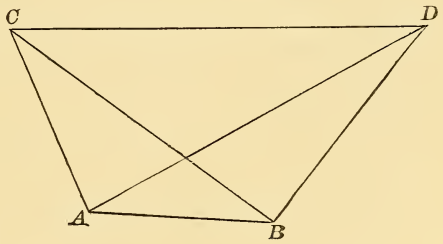
24. Two fixed objects, A and B, and a ship, were all observed to be in a line bearing N. 33° 15′ E. The ship then sailed northwest 10 miles, when the bearing of A was found to be east, and that of B northeast. Find the distance from A to B.

25. The perimeter of a triangle is 400 feet, and the angles are 65° 15′, 75° 30′, and 39° 15′, respectively. Find the sides.

26. Express the area of a triangle, in terms of the base b and the two adjacent angles, θ and θ'.

27. Two sides of a parallelogram are 90 feet and 110 feet, and one of the diagonals 120 feet. Find the other diagonal, and the angles of the parallelogram.

28. Given $AB = 100$ rods,
angle $BAD = 32°$,
$BAC = 98°$,
$ABC = 37°$,
$ABD = 118°$;



to find the distance between C and D.

29. Find the perpendiculars let fall from the vertices of a triangle upon the opposite sides, when $a = 25$, $b = 30$, $c = 35$.

30. What is the angle of depression from the top of a mountain 3 miles high, the earth's radius being 3962 miles?

31. A point of land was observed by a ship at sea to bear S. 11° 15′ E.; and after sailing northeast 12 miles, it was found to bear S. 33° 45′ E. How far was the point from the ship at the last observation?

32. A side of the base of a square pyramid is 200 feet, and each edge is 150 feet. Find the slope of each face.

33. From the top of a mountain, 3 miles high, the angle of depression of the remotest visible point of the earth's surface is

$2° 13' 27''$. Find the radius of the earth and the utmost distance from which the mountain is visible.

34. Two sides of a triangle are 30.8 and 54.12, and the angle opposite the latter is $36° 42' 11''$. How many triangles? Solve.

35. Two sides of a triangle are 600 and 250, and the angle opposite the latter is $42° 12'$. Solve the triangle.

36. If $\cos\theta = \dfrac{\cos\theta' - \epsilon}{1 - \epsilon\cos\theta'}$, show that $\tan\frac{1}{2}\theta = \sqrt{\dfrac{1+\epsilon}{1-\epsilon}}\tan\frac{1}{2}\theta'$.

37. The angles of a triangle are as 3, 4, and 5, and the least side is 10. Find the other sides.

38. The radius of the earth being 3962 miles, what is the length of $1°$ of the meridian?

39. At three points in the same horizontal straight line the angles of elevation of an object were found to be $36° 50'$, $21° 24'$, and $14°$, the middle station being 34 feet from each of the others. Required the height of the object.

40. If r, r', r'', r''', denote the radii of the inscribed and escribed circles of a triangle, show that

$$\tan^2\frac{A}{2} = \frac{rr'}{r''r'''}.$$

41. Three circles, whose radii are r, r', r'', touch one another; show that the radius of the circle passing through the three points of contact is

$$\sqrt{\frac{rr'r''}{r + r' + r''}}.$$

42. The sides of a triangle are in arithmetical progression, and their common difference is 2 inches. If the area is $3\sqrt{15}$ square inches, find the sides.

43. The area of a triangle is 84 square inches, and two of its sides are 15 and 13 inches. Find the third side.

44. Given the vertical angle, the base, and the difference between the two sides of the triangle. Find the other angles.

CHAPTER VII.

SPHERICAL TRIANGLES.

75. The object of Spherical Trigonometry is the solution of spherical triangles.

The sides as well as the angles of a spherical triangle are expressed in degrees, minutes, and seconds.

Only those spherical triangles whose sides are, each, less than 180° will here be considered.

76. Let ABC (Fig. 23) be a spherical triangle traced upon the surface of a sphere, whose centre is O, and radii be drawn from O to the three vertices. A tri-edral angle will be formed whose plane angles AOB, AOC, and BOC, are respectively equal to the sides c, b, and a, since the latter are the arcs that measure those angles.

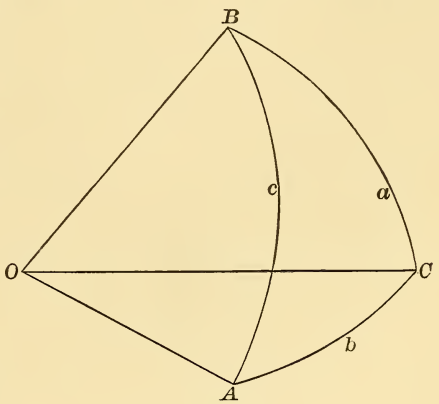


Fig. 23.

The di-edral angles AOB–AOC, AOB–BOC, and AOC–BOC are equal to the angles A, B, and C, respectively. The planes which form the tri-edral angle AOB–AOC–BOC always cut the surface of the sphere in arcs of great circles, thus tracing upon the surface of the sphere a spherical triangle, whose sides are arcs of great circles.

77. *Definitions.*

A spherical right triangle has one, two, or three, right angles.

When it has two right angles, the triangle is called *bi-rectangular*.

When it has three right angles, the triangle is *tri-rectangular*.

A spherical triangle having one side = 90° is called *quadrantal.* When it has two sides, each = 90°, it is *bi-quadrantal;* and when three sides, each = 90°, it is *tri-quadrantal* or *tri-rectangular.*

78. *Relations between the angles and the sides of a spherical triangle.*

Let ABC (Fig. 24) be a spherical triangle traced on the surface of a sphere whose centre is O, and with a radius taken equal to *unity.* Let the sides b and c, each, be less than 90°.

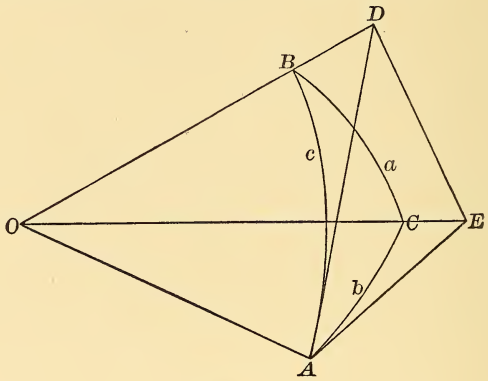


Fig. 24.

Draw the tangents AD and AE, meeting the radii OB and OC prolonged, in the points D and E, respectively.

We have $AD = \tan c$, $OD = \sec c$, $AE = \tan b$, $OE = \sec b$.

Also, the angle $DAE = A$, and the angle $DOE = a$.

The plane triangles DAE and DOE give

$$(1)\quad \overline{DE}^2 = \overline{AD}^2 + \overline{AE}^2 - 2\,\overline{AD}\cdot\overline{AE}\cos DAE,$$

$$(2)\quad \overline{DE}^2 = \overline{OD}^2 + \overline{OE}^2 - 2\,\overline{OD}\cdot\overline{OE}\cos DOE;$$

from which we obtain

$$(3)\quad 2\,\overline{OD}\cdot\overline{OE}\cos DOE = (\overline{OD}^2 - \overline{AD}^2) + (\overline{OE}^2 - \overline{AE}^2) + 2\,\overline{AD}\cdot\overline{AE}\cos DAE.$$

Since AD and AE are tangents at the point A, the extremity of the radius OA, the angles DAO and EAO are right angles, and (3) becomes

$$(4)\quad 2\,\overline{OD}\cdot\overline{OE}\cos DOE = 2\,\overline{OA}^2 + 2\,\overline{AD}\cdot\overline{AE}\cos DAE.$$

Replacing the various quantities in (4) by their values, we have

$$(5)\quad \sec b \sec c \cos a = 1 + \tan b \tan c \cos A,$$

whence multiplying (5) by $\cos b \cos c$, we obtain

$$(6)\quad \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

79. We have supposed the sides b and c, each, less than 90°, but we shall now show that whatever values b and c may have between 0 and 180°, (6) holds good.

Suppose $c > 90°$, and $b < 90°$, and prolong the arcs BA and BC (Fig. 25) until they meet at B'. Let $AB' = c'$, and $CB' = a'$; then the triangle $AB'C$ will give

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos B'AC,$$

for the sides c' and b are each less than 90°.

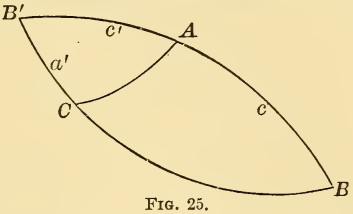


FIG. 25.

Replacing a', c', and $B'AC$ by their values $180° - a$, $180° - c$, and $180° - A$, we obtain as before

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Again, suppose $b > 90°$ and $c > 90°$.

Prolong the sides AB and AC (Fig. 26) until they meet at A'. Let $A'C = b'$, and $A'B = c'$; then the triangle $A'BC$ will give

$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Replacing b', c', and A' by their values $180° - b$, $180° - c$, and A, we obtain as before

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We conclude, therefore, that whatever values the sides may have between the limits of 0 and 180°, the formula holds.

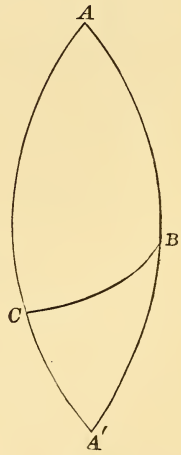


FIG. 26.

Thus by a change of letters we have three equations,

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A, \\ \cos b &= \cos a \cos c + \sin a \sin c \cos B, \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C, \end{aligned}\right\} \quad [47]$$

which may be regarded as the fundamental formulæ of Spherical Trigonometry.

80. *Relations existing between a side and the three angles of a spherical triangle.*

Let ABC be a spherical triangle whose polar is the triangle $A'B'C'$ (Fig. 27).

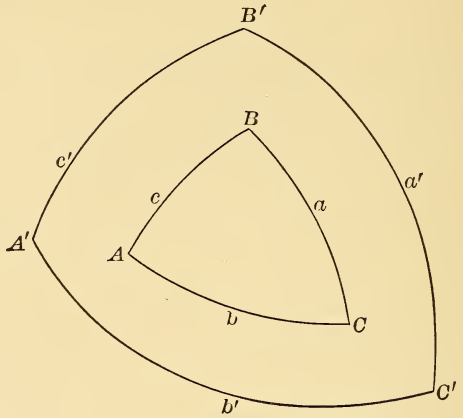


Fig. 27.

From the geometric principles of polar triangles, we have

$$A' = 180° - a, \qquad a' = 180° - A,$$
$$B' = 180° - b, \qquad b' = 180° - B,$$
$$C' = 180° - c, \qquad c' = 180° - C.$$

By [47],

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Replacing the values of A', B', C', a', b', and c', we have

$$\cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a),$$

or

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

This equation, after changing the signs, and two others similar to it, constitute the following group,

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B &= -\cos A \cos C + \sin A \sin C \cos b, \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned}\right\} \qquad [48]$$

81. *Relations existing between two sides and the angles opposite.*

To obtain one relation between the sides a and b, and the angles A and B, is accomplished in a very simple manner by introducing into formulæ [47] the sines of the angles A, B, and C, in place of the cosines.

The first of [47] gives

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

from which

$$\sin^2 A = 1 - \cos^2 A = \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

and, dividing both members by $\sin^2 a$, we have

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

This value of $\frac{\sin^2 A}{\sin^2 a}$ does not change when we permute the letters a, b, and c. It follows, then, that the other two formulæ of [47] will give the same value for $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$, precisely as for $\frac{\sin^2 A}{\sin^2 a}$. Therefore, since the angles and the sides of a triangle are, each, less than 180°, their sines are positive, and we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \qquad [49]$$

Therefore, *in a spherical triangle the sines of the angles are proportional to the sines of the opposite sides.*

82. *Relations existing between the three sides and two of the angles of a spherical triangle.*

If the value of the $\cos c$ given in the third equation of [47] be substituted in the first of those equations, we shall have

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A,$$

which by transposition, putting for $\cos^2 b$ its value $1 - \sin^2 b$, and dividing by $\sin b$, becomes

$$\cos a \sin b - \sin a \cos b \cos C = \sin c \cos A.$$

By permutation of the letters, five other similar formulæ may be obtained, so that we have

$$\left.\begin{aligned}
&\cos a \sin b - \sin a \cos b \cos C = \sin c \cos A,\\
&\cos b \sin a - \sin b \cos a \cos C = \sin c \cos B,\\
&\cos b \sin c - \sin b \cos c \cos A = \sin a \cos B,\\
&\cos c \sin b - \sin c \cos b \cos A = \sin a \cos C,\\
&\cos c \sin a - \sin c \cos a \cos B = \sin b \cos C,\\
&\cos a \sin c - \sin a \cos c \cos B = \sin b \cos A.
\end{aligned}\right\} \quad [50]$$

$\sin a$, $\sin b$, and $\sin c$, by [49], may be replaced by $\sin A$, $\sin B$, and $\sin C$, and thus we shall obtain

$$\left.\begin{aligned}
&\cos a \sin B - \cos b \cos C \sin A = \cos A \sin C,\\
&\cos b \sin A - \cos a \cos C \sin B = \cos B \sin C,\\
&\cos b \sin C - \cos c \cos A \sin B = \cos B \sin A,\\
&\cos c \sin B - \cos b \cos A \sin C = \cos C \sin A,\\
&\cos c \sin A - \cos a \cos B \sin C = \cos C \sin B,\\
&\cos a \sin C - \cos c \cos B \sin A = \cos A \sin B.
\end{aligned}\right\} \quad [51]$$

83. *Relations existing between two sides, the angle included between those two sides, and the angle opposite one of them.*

If we divide the first equation of [50] by

$$\sin a = \frac{\sin c \sin A}{\sin C}, \quad [49],$$

member by member, the first of the relations sought will be found. We may obtain the others by permutation of the letters.

$$\left.\begin{aligned}
&\cot a \sin b - \cot A \sin C = \cos b \cos C,\\
&\cot b \sin a - \cot B \sin C = \cos a \cos C,\\
&\cot b \sin c - \cot B \sin A = \cos c \cos A,\\
&\cot c \sin b - \cot C \sin A = \cos b \cos A,\\
&\cot c \sin a - \cot C \sin B = \cos a \cos B,\\
&\cot a \sin c - \cot A \sin B = \cos c \cos B.
\end{aligned}\right\} \quad [52]$$

84. *Formulæ relating to spherical right triangles, when there is but one right angle.*

If, in the first equation of [47], in two of the equations of [49], in the first, third, sixth, and fourth of [52], and in all of [48], each of which contains the angle A, we suppose the angle A equal to 90°, then the following formulæ, peculiar to spherical right triangles, will result.

$$\left.\begin{array}{rll}
\text{From [47],} & (1) & \cos a = \cos b \cos c, \\
\text{[49],} & (2) & \sin b = \sin a \sin B, \\
& (3) & \sin c = \sin a \sin C, \\
\text{[52],} & (4) & \tan b = \tan a \cos C, \\
& (5) & \tan b = \sin c \tan B, \\
& (6) & \tan c = \tan a \cos B, \\
& (7) & \tan c = \sin b \tan C, \\
\text{[48],} & (8) & \cos a = \cot B \cot C, \\
& (9) & \cos B = \cos b \sin C, \\
& (10) & \cos C = \cos c \sin B.
\end{array}\right\} \quad [53]$$

85. *General formulæ adapted to logarithmic computation.*

In order to render [47] and [48] suitable for logarithmic computation, let us resume the fundamental formulæ [47],

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$(1) \qquad \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

$$(2) \quad 1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

$$(3) \quad 1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

But by substituting the values of $1 - \cos A$, and $1 + \cos A$, as found by [26] and [27], (Pl. Tr.), we shall find that (2) becomes

$$(4) \quad \sin \tfrac{1}{2} A = \sqrt{\frac{\sin b \sin c + \cos b \cos c - \cos a}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{\cos(b-c) - \cos a}{2 \sin b \sin c}} \qquad \text{(Art. 52.)}$$

$$= \sqrt{\frac{\sin \tfrac{1}{2}(a - b + c) \sin \tfrac{1}{2}(a + b - c)}{\sin b \sin c}} \qquad \text{(Art. 65, Pl. Tr.)}$$

$$= \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}}.$$

Operating upon (3) in a similar manner, we find

$$(5)\quad \cos\tfrac{1}{2}A=\sqrt{\frac{\cos a-\cos b\cos c+\sin b\sin c}{2\sin b\sin c}}=\sqrt{\frac{\cos a-\cos(b+c)}{2\sin b\sin c}}$$

$$=\sqrt{\frac{\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(b+c-a)}{\sin b\sin c}}$$

$$=\sqrt{\frac{\sin s\sin(s-c)}{\sin b\sin c}}.$$

Now, if (4) be divided by (5), the result will be

$$(6)\quad \tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

By a simple change of letters, analogous expressions for the sine, cosine, and tangent, of the angles $\frac{1}{2}B$ and $\frac{1}{2}C$, may be found.

The three systems are as follows:

$$\left.\begin{aligned}\sin\tfrac{1}{2}A&=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}},\\ \sin\tfrac{1}{2}B&=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}},\\ \sin\tfrac{1}{2}C&=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}.\end{aligned}\right\}\quad[54]$$

$$\left.\begin{aligned}\cos\tfrac{1}{2}A&=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}},\\ \cos\tfrac{1}{2}B&=\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}},\\ \cos\tfrac{1}{2}C&=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.\end{aligned}\right\}\quad[55]$$

$$\left.\begin{aligned}\tan\tfrac{1}{2}A&=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}},\\ \tan\tfrac{1}{2}B&=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}},\\ \tan\tfrac{1}{2}C&=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.\end{aligned}\right\}\quad[56]$$

The foregoing formulæ express the sine, cosine, and tangent of one half an angle of a spherical triangle in terms of sides.

The radical quantities in each one of the formulæ must be taken positively, because $\frac{1}{2}A$, $\frac{1}{2}B$, and $\frac{1}{2}C$ are each less than 90°, and therefore their trigonometrical functions are positive.

86. *The sine, cosine, and tangent of a half side in terms of the angles of a spherical triangle.*

From the first of [48] we have

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\therefore (1)\quad 1 - \cos a = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C} = -\frac{\cos A + \cos(B+C)}{\sin B \sin C};$$

$$(2)\quad 1 + \cos a = 1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C} = \frac{\cos A + \cos(B-C)}{\sin B \sin C};$$

$$\therefore (3)\quad \sin\tfrac{1}{2}a = \sqrt{-\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B \sin C}};$$

$$(4)\quad \cos\tfrac{1}{2}a = \sqrt{\frac{\cos\frac{1}{2}(A-B+C)\cos\frac{1}{2}(A+B-C)}{\sin B \sin C}}.$$

Now, let $A+B+C=2S$; then $B+C-A=2(S-A)$;

$A-B+C=2(S-B)$; and $A+B-C=2(S-C)$;

and substituting in (3) and (4), they become

$$(5)\quad \sin\tfrac{1}{2}a = \sqrt{-\frac{\cos S \cos(S-A)}{\sin B \sin C}},$$

$$(6)\quad \cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}.$$

Divide (5) by (6) and obtain

$$(7)\quad \tan\tfrac{1}{2}a = \sqrt{-\frac{\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

The functions of $\frac{1}{2}b$ and of $\frac{1}{2}c$ may be obtained from (5), (6), and (7), by a simple change of letters.

The three systems are as follows:

$$\left.\begin{aligned} \sin\tfrac{1}{2}a &= \sqrt{-\frac{\cos S\cos(S-A)}{\sin B\sin C}}, \\ \sin\tfrac{1}{2}b &= \sqrt{-\frac{\cos S\cos(S-B)}{\sin A\sin C}}, \\ \sin\tfrac{1}{2}c &= \sqrt{-\frac{\cos S\cos(S-C)}{\sin A\sin B}}. \end{aligned}\right\} \quad [57]$$

$$\left.\begin{aligned} \cos\tfrac{1}{2}a &= \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}, \\ \cos\tfrac{1}{2}b &= \sqrt{\frac{\cos(S-A)\cos(S-C)}{\sin A\sin C}}, \\ \cos\tfrac{1}{2}c &= \sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}}. \end{aligned}\right\} \quad [58]$$

$$\left.\begin{aligned} \tan\tfrac{1}{2}a &= \sqrt{-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}, \\ \tan\tfrac{1}{2}b &= \sqrt{-\frac{\cos S\cos(S-B)}{\cos(S-A)\cos(S-C)}}, \\ \tan\tfrac{1}{2}c &= \sqrt{-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}. \end{aligned}\right\} \quad [59]$$

Remark I. The positive sign must be given to the radicals of [57], [58], and [59], because $\frac{1}{2}a$, $\frac{1}{2}b$, and $\frac{1}{2}c$ are, each, less than 90°.

Remark II. The functions of $\frac{1}{2}a$, $\frac{1}{2}b$, and $\frac{1}{2}c$, in [57] and [59], are *real* quantities. For since the sum of the angles of a spherical triangle is greater than 180°, and less than six right angles, then S, or $\frac{1}{2}(A+B+C)$, in [57] and [59], is greater than 90°, and less than three right angles. Therefore the $\cos S$ is either in the second or third quadrant, and is negative. The quantities under the radical sign are, therefore, positive.

It is easily shown that $\cos(S-A)$, $\cos(S-B)$, and $\cos(S-C)$ are all positive and do not change the result.

87. *Formulæ of Delambre.*

From formulæ [54] and [55], we obtain

(1)
$$\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B=\frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}=\frac{\sin(s-b)}{\sin c}\cos\tfrac{1}{2}C,$$

(2)
$$\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B=\frac{\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}=\frac{\sin(s-a)}{\sin c}\cos\tfrac{1}{2}C,$$

(3)
$$\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B=\frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}=\frac{\sin s}{\sin c}\sin\tfrac{1}{2}C,$$

(4)
$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B=\frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}=\frac{\sin(s-c)}{\sin c}\sin\tfrac{1}{2}C.$$

From (1) and (2) by addition and subtraction, we obtain

(5) $$\frac{\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\pm\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B}{\cos\tfrac{1}{2}C}=\frac{\sin(s-b)\pm\sin(s-a)}{\sin c};$$

and from (3) and (4), we obtain

(6) $$\frac{\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\mp\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B}{\sin\tfrac{1}{2}C}=\frac{\sin s\mp\sin(s-c)}{\sin c}.$$

By Art. 52,

(7) $$\sin(s-b)+\sin(s-a)=2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(a-b),$$

(8) $$\sin(s-b)-\sin(s-a)=2\cos\tfrac{1}{2}c\sin\tfrac{1}{2}(a-b),$$

(9) $$\sin s+\sin(s-c)=2\cos\tfrac{1}{2}c\sin\tfrac{1}{2}(a+b),$$

(10) $$\sin s-\sin(s-c)=2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(a+b),$$

also $$\sin c=2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c.$$

∴ substituting in (5) and (6) the values found in (7), (8), (9), (10), and that of $\sin c$, we have

$$\left.\begin{aligned}
(1)\quad & \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}, \\
(2)\quad & \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}, \\
(3)\quad & \frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}, \\
(4)\quad & \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} = \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.
\end{aligned}\right\} \qquad [60]$$

Remark. These formulæ were discovered by Delambre, in 1807, and published in *Connaissance des Temps* for 1809 (p. 443). Gauss, to whom they are sometimes attributed, did not publish them until two years later, in his work, *Theoria motus corporum cœlestium.*

88. *Napier's Analogies.*

If (1) of [60] be divided by (3); (2) by (4); (4) by (3); and (2) by (1), we shall obtain the formulæ that are known as *Napier's Analogies.*

They are as follows:

$$\left.\begin{aligned}
(1)\quad & \tan\tfrac{1}{2}(A+B) = \cot\tfrac{1}{2}C\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}, \\
(2)\quad & \tan\tfrac{1}{2}(A-B) = \cot\tfrac{1}{2}C\,\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}, \\
(3)\quad & \tan\tfrac{1}{2}(a+b) = \tan\tfrac{1}{2}c\,\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}, \\
(4)\quad & \tan\tfrac{1}{2}(a-b) = \tan\tfrac{1}{2}c\,\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}.
\end{aligned}\right\} \qquad [61]$$

CHAPTER VIII.

SOLUTION OF SPHERICAL RIGHT TRIANGLES.

89. The formulæ for the solution of spherical right triangles are those of [53], Art. 84.

From formula (1), $\cos a = \cos b \cos c$; it follows that either all the cosines are positive, or but one is positive.

Therefore, *in a right triangle, either all the sides are less than quadrants, or one side is less than a quadrant, and the other two sides are greater than quadrants.*

From (5) and (7) of [53], it follows that $\tan b$ and $\tan B$ have the same sign, and are, therefore, either both greater than 90°, or both less than 90°. When such is the case, b and B are of the *same species.* The same is true of c and C. When one is greater than 90°, and the other less than 90°, they are then of *different species.*

90. The ten equations of [53] constitute but six cases:

1. Given the two sides; to find the other parts.
2. Given a side and its opposite angle; to find the other parts.
3. Given the two angles; to find the other parts.
4. Given the hypothenuse and one side; to find the other parts.
5. Given the hypothenuse and an angle; to find the other parts.
6. Given a side and an adjacent angle; to find the other parts.

91. *Napier's Circular Parts.* Two rules, called *Napier's Rules of Circular Parts* of a spherical triangle, include all the possible cases.

The *circular parts* of a spherical triangle are five: the two sides b and c, the *complements* of the angles B and C, and the *complement* of the hypothenuse a. The complements of B, C, and a are generally written co. B, co. C, and co. a. The right angle is always excluded.

An examination of Fig. 28 shows that if any three parts be taken, as b, c, and co. B, all are adjacent; but if b, c, and co. a be taken, then one is separated from the other two by the remaining

parts co. B and co. C. No other arrangements of the five parts, when taken by threes, can be made. The three are either adjacent, or one is separated from the other two by intervening parts.

When the parts are adjacent, as co. C, co. a, and co. B, then one is called the *middle part*, and the other two the *adjacent parts*.

When the parts are separated, as co. a, co. B, and b, one, b, is the *middle part*, and the other two, co. a and co. B, are called the *opposite parts*.

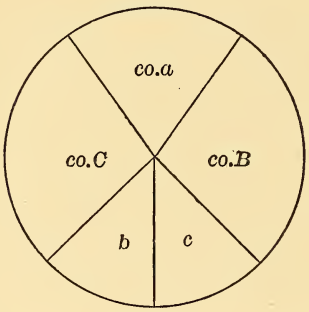


FIG. 28.

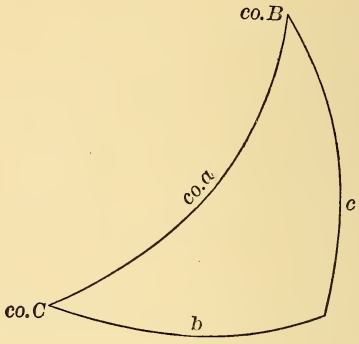


FIG. 29.

Let Fig. 29 be a spherical right triangle, in which b and c are taken as before; but the angle B is represented by co. B, or $90°-B$, the angle C by co. C, or $90°-C$, and the hypothenuse a by co. a, or $90°-a$.

A comparison of Figs. 28 and 29 will show that *Napier's Rules* apply in either case.

92. *Napier's Rules.*

1. *The sine of the middle part equals the product of the tangents of the adjacent parts.*

2. *The sine of the middle part equals the product of the cosines of the opposite parts.*

93. That *Napier's Rules, the formulæ* of [53], Art. 84, and *the six cases* of Art. 90, all agree, we shall now show.

Formula (8) of [53] is

$$\cos a = \cot B \cot C.$$

By Napier's first rule,

$$\sin(\text{co. } a) = \tan(\text{co. } B)\tan(\text{co. } C).$$

Replacing co. a, co. B, and co. C by their values, the result becomes

$$\sin(90°-a) = \tan(90°-B)\tan(90°-C),$$

or $$\cos a = \cot B \cot C.$$

Again, formula (9) of [53] is

$$\cos B = \cos b \sin C.$$

By Napier's second rule,

$$\sin(\text{co. } B) = \cos b \cos(\text{co. } C),$$

or $$\sin(90°-B) = \cos b \cos(90°-C),$$

or $$\cos B = \cos b \sin C.$$

Six cases are to be considered.

94. Given b and c; to find the other parts.

From (5), [53], $\tan b = \sin c \tan B$; $\therefore \tan B = \dfrac{\tan b}{\sin c}$.

(7), $\tan c = \sin b \tan C$; $\therefore \tan C = \dfrac{\tan c}{\sin b}$.

(1), $\cos a = \cos b \cos c$.

It is clear that no ambiguity exists in the solutions, since not one of the quantities required is to be found from its sine.

95. Given b and B; to find the other parts.

From

(2), [53], $\sin b = \sin a \sin B$; $\therefore \sin a = \dfrac{\sin b}{\sin B}$.

(9), $\cos B = \cos b \sin C$; $\therefore \sin C = \dfrac{\cos B}{\cos b}$.

(5), $\tan b = \sin c \tan B$; $\therefore \sin c = \dfrac{\tan b}{\tan B}$.

In this case there is nothing to decide whether a, C, and c should be greater or less than 90°; therefore the solution remains ambiguous, as the parts are determined from their sines. It is the only ambiguous case that occurs in spherical right triangles.

Let ABC (Fig. 30), right-angled at A, be a triangle

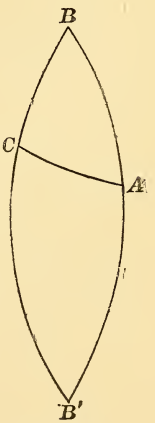


FIG. 30.

satisfying the given conditions. Prolong BA and BC until they meet at B'. Then the right triangle $AB'C$ also satisfies the given conditions; for it has the angle B' equal to B, the angle A a right angle, and the side AC the same as in ABC.

96. Given B and C; to find the other parts.

From (8), [53], $\cos a = \cot B \cot C$:

(9), $\cos B = \cos b \sin C$; $\therefore \cos b = \dfrac{\cos B}{\sin C}$.

(10), $\cos C = \cos c \sin B$; $\therefore \cos c = \dfrac{\cos C}{\sin B}$.

There is no ambiguity in this case.

97. Given hypothenuse a and a side b; to find the other parts.

From (1), [53], $\cos a = \cos b \cos c$; $\therefore \cos c = \dfrac{\cos a}{\cos b}$.

(2), $\sin b = \sin a \sin B$; $\therefore \sin B = \dfrac{\sin b}{\sin a}$.

(4), $\tan b = \tan a \cos C$; $\therefore \cos C = \dfrac{\tan b}{\tan a}$.

There can be no ambiguity in this case, except in the value of B, but this is removed by the consideration that B and b are always of the same species.

98. Given hypothenuse a and an angle C; to find the other parts.

From (8), [53], $\cos a = \cot B \cot C$; $\therefore \cot B = \dfrac{\cot a}{\cot C}$.

(4), $\tan b = \tan a \cos C$.

(3), $\sin c = \sin a \sin C$.

There is no ambiguity in (3), when it is remembered that c and C are of the same species.

99. Given a side b and an adjacent angle C; to find the other parts.

From (9), [53], $\cos B = \cos b \sin C$.

(7), $\tan c = \sin b \tan C$.

(4), $\tan b = \tan a \cos C$; $\therefore \tan a = \dfrac{\tan b}{\cos C}$.

There is no ambiguity in this case.

100. *Isosceles and Quadrantal Triangles.*

An *isosceles triangle* is readily solved by dividing it into two right triangles by a perpendicular from the angle included by the equal sides, then applying formulæ [53].

A *quadrantal triangle* which is the polar triangle of a right triangle, has one of its sides equal to 90°. When such a triangle appears, it is easily solved by means of its polar triangle.

101. Examples.

SET VII.

Isosceles Triangles.

1. In a spherical right triangle, $c = 145°$, and $A = 23° 28'$, C being the right angle. Find the other parts.

2. Given $c = 32° 34'$, and $A = 44° 44'$. Find B, a, and b.

3. Given $a = 141° 11'$, and $c = 127° 12'$. Find A, B, and b.

4. Given $a = 35° 44'$, and $A = 37° 28'$. Find B, b, and c. This problem gives two triangles.

5. Given $a = 118° 54'$, and $B = 12° 19'$. Find A, b, and c.

6. Given $A = 91° 11'$, and $B = 111° 11'$. Find a, b, and c.

7. Given $a = 1°$, and $B = 100°$. Find A, B, and c.

8. Given $A = 23° 28'$, and $b = 49° 17'$. Find B, a, and c.

9. Given $a = 37° 48'$, and $c = 66° 32'$. Find A, B, and b.

10. Given $a = 59° 38' 27''$, and $b = 48° 24' 16''$. Find A, B, and c.

11. Given $B = 111° 14' 37''$, and $b = 121° 26' 25''$. Find A, a, and c.

Quadrantal Triangles.

12. Given $c = 90°$, $A = 54° 43'$, and $B = 42° 12'$. Find the other parts.

13. Given $c = 90°$, $A = 112° 2' 9''$, and $b = 67° 3' 14''$. Find the other parts.

14. Given $c = 90°$, $a = 22° 53' 30''$, and $b = 51° 4' 35''$. Find the other parts.

15. Given the quadrantal side AB, the angles A and B; to find the other parts.

16. Prove $\sin^2\frac{1}{2}c = \sin^2\frac{1}{2}a\cos^2\frac{1}{2}b + \cos^2\frac{1}{2}a\sin^2\frac{1}{2}b$. C being 90°.

17. Prove $\tan^2\frac{1}{2}b = \tan\frac{1}{2}(c+a)\tan\frac{1}{2}(c-a)$.

18. Prove $\tan^2\frac{1}{2}A = \sin(c-b)\operatorname{cosec}(c+b)$.

19. Prove $\tan^2\frac{1}{2}c = -\cos(A+B)\sec(A-B)$.

20. Prove $\tan^2(45°-\frac{1}{2}b) = \sin(A-a)\operatorname{cosec}(A+a)$.

21. In a right triangle show that

$$\sin(c-b) = 2\sin^2\tfrac{1}{2}A\cos b\sin c.$$

22. $C = 90°$. Show that $\sin a\cos b = \tan\frac{1}{2}A\sin(b+c)$.

23. The equal sides of an isosceles triangle are, each, 45°, and the angle included is 95°. Find the other parts.

24. In a right triangle, if θ be the length of the arc drawn from C, perpendicular to the hypothenuse c, show that

$$\cot\theta = \sqrt{\cot^2 a + \cot^2 b}.$$

25. a is one side of an equilateral triangle. Find the angle A.

26. Show that $\dfrac{\cos a}{\cos b} = \dfrac{\sin 2A}{\sin 2B}$.

27. Required the diedral angles made by the faces of the regular polyedrons.

28. Show that $\sin(c-a) = \sin b\cos a\tan\frac{1}{2}B$.

29. If $A = 36°$, $B = 60°$, and $C = 90°$, show that

$$a+b+c = 90°.$$

30. C being a right angle, show that

$$\sin A\sin 2b = \sin c\sin 2B.$$

CHAPTER IX.

SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

102. We shall consider only those triangles each of whose parts is less than 180°, keeping in mind always the following principles:

(1) *The greater side is opposite the greater angle, and conversely,*

(2) *Each side is less than the sum of the other two.*

(3) *The sum* $a+b+c$ *is less than* 360°.

(4) *The sum* $A+B+C$ *is greater than* 180°.

(5) *If* $A+B+C>180°$, *then* $A>180°-B-C$,
$B>180°-A-C$,
$C>180°-A-B$.

(6) *A side differing more from* 90° *than another side is in the same quadrant as its opposite angle.*

(7) *An angle differing more from* 90° *than another angle is in the same quadrant as its opposite side.*

(8) *Two sides at least are in the same quadrants as their opposite angles respectively.*

(9) *The sum of two sides is* $>$, $=$, *or* $<180°$, *according as the sum of the two opposite angles is* $>$, $=$, *or* $<180°$.

103. Case I. *Given the three sides; to find the angles.*

Formulæ [54], [55], and [56], which express the sine, cosine, and tangent of one-half an angle in terms of the sides of a spherical triangle, may be used. The preference is to be given to [56].

Example. Given the three sides,

$a = 113°\ 2'\ 56''.64,$
$b = 82°\ 39'\ 28''.40,$
$c = 74°\ 54'\ 31''.06;$

to find the angles A, B, and C.

Solution :

$s = \frac{1}{2}(a+b+c) = 135° 18' 28''.05$ $\quad \log \sin s = \bar{1}.847139$

$s - a = 22° 15' 31''.41$ $\quad \log \sin(s-a) = \bar{1}.578398$

$s - b = 52° 38' 59''.65$ $\quad \log \sin(s-b) = \bar{1}.900336$

$s - c = 60° 23' 56''.99$ $\quad \log \sin(s-c) = \bar{1}.939264.$

$$(1)\quad \tan \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

$$(2)\quad \tan \tfrac{1}{2}B = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}}.$$

$\log \sin(s-b) = \bar{1}.900336$ $\quad \log \sin(s-a) = \bar{1}.578398$

$\log \sin(s-c) = \bar{1}.939264$ $\quad \log \sin(s-c) = \bar{1}.939264$

$\operatorname{colog} \sin s = 0.152861$ $\quad \operatorname{colog} \sin s = 0.152861$

$\operatorname{colog} \sin(s-a) = 0.421602$ $\quad \operatorname{colog} \sin(s-b) = 0.099664$

$2\overline{)0.414063}$ $\quad 2\overline{)\bar{1}.770187}$

$\log \tan \frac{1}{2}A = 0.207031$ $\quad \log \tan \frac{1}{2}B = \bar{1}.885093$

$\frac{1}{2}A = 58° 10' 1''.10.$ $\quad \frac{1}{2}B = 37° 30' 25''.8.$

$\therefore A = 116° 20' 2''.20.$ $\quad \therefore B = 75° 0' 51''.6.$

$$(3)\quad \tan \tfrac{1}{2}C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}.$$

$\log \sin(s-a) = \bar{1}.578398$

$\log \sin(s-b) = \bar{1}.900336$

$\operatorname{colog} \sin s = 0.152861$

$\operatorname{colog} \sin(s-c) = 0.060736$

$2\overline{)\bar{1}.692331}$

$\log \tan \frac{1}{2}C = \bar{1}.846166$

$\frac{1}{2}C = 35° 3' 29''.58$

$\therefore C = 70° 6' 59''.16.$

104. Case II. *Given the three angles; to find the sides.*

Formulæ [57], [58], and [59] are applicable. But the three formulæ of [59] are generally to be preferred.

105. Case III. *Given two sides and the included angle; to find the other parts.*

We obtain the simplest solution of this problem by means of Napier's Analogies, formulæ [61], Art. 88.

If the given parts be a, b, and C, we compute at once A and B by (1) and (2) of [61]; then c by (3) or (4).

Example. Given $a = 113°\ 2'\ 56''.64$,
$b = 82°\ 39'\ 28''.40$,
$C = 138°\ 50'\ 13''.69$; to find A, B, and c.

Solution:

$\frac{1}{2}(a-b) = 15°\ 11'\ 44''.12$ $\quad \log\sin\frac{1}{2}(a-b) = \bar{1}.418492$
$\frac{1}{2}(a+b) = 97°\ 51'\ 12''.52$ $\quad \log\cos\frac{1}{2}(a-b) = \bar{1}.984544$
$\frac{1}{2}C = 69°\ 25'\ 6''.845$ $\quad \log\sin\frac{1}{2}(a+b) = \bar{1}.995907$
$\log\cos\frac{1}{2}(a+b) = \bar{1}.135578$
$\log\cot\frac{1}{2}C = \bar{1}.574616$

$$(1)\quad \tan\tfrac{1}{2}(A+B) = \cot\tfrac{1}{2}C\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}.$$

$$(2)\quad \tan\tfrac{1}{2}(A-B) = \cot\tfrac{1}{2}C\,\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$$

$\log\cot\frac{1}{2}C = \bar{1}.574616$	$\log\cot\frac{1}{2}C = \bar{1}.574616$
$\log\cos\frac{1}{2}(a-b) = \bar{1}.984544$	$\log\sin\frac{1}{2}(a-b) = \bar{1}.418492$
$\operatorname{colog}\cos\frac{1}{2}(a+b) = 0.864421$	$\operatorname{colog}\sin\frac{1}{2}(a+b) = 0.004093$
$\log\tan\frac{1}{2}(A+B) = 0.423581$	$\log\tan\frac{1}{2}(A-B) = \bar{2}.997201$

$\therefore\ \frac{1}{2}(A+B) = 110°\ 39'\ 35''.29.$ $\quad \therefore\ \frac{1}{2}(A-B) = 5°\ 40'\ 26''.91.$

$$\tfrac{1}{2}(A+B) = 110°\ 39'\ 35''.29,$$
$$\tfrac{1}{2}(A-B) = 5°\ 40'\ 26''.91.$$

$$\therefore\ A = 116°\ 20'\ 2''.20,$$
$$B = 104°\ 59'\ 8''.38.$$

$\log\cos\frac{1}{2}(A+B) = \bar{1}.547551.$ $\quad \log\cos\frac{1}{2}(A-B) = \bar{1}.997867.$

$$(3)\quad \tan\tfrac{1}{2}c = \tan\tfrac{1}{2}(a+b)\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}.$$

$$\log\tan\tfrac{1}{2}(a+b) = \begin{cases} \log\sin\frac{1}{2}(a+b) = \bar{1}.995907 \\ \operatorname{colog}\cos\frac{1}{2}(a+b) = 0.864421 \end{cases}$$

$$\log\cos\tfrac{1}{2}(A+B) = \bar{1}.547551$$

$$\operatorname{colog}\cos\tfrac{1}{2}(A-B) = 0.002133$$

$$\log\tan\tfrac{1}{2}c = 0.410012$$

$$\tfrac{1}{2}c = 68^\circ\,44'\,32''.30.$$

$$\therefore\ c = 137^\circ\,29'\ \ 4''.6.$$

106. Case IV. *Two angles and the included side given; to find the other parts.*

If the given parts be A, B, and c, we compute a and b by formulæ (3) and (4), and C by (1) or (2), of Napier's Analogies.

107. Case V. *Given two sides and an angle opposite one of them; to find the other parts.*

If the given parts be a, b, and A, we compute B at once by the formula $\frac{\sin B}{\sin b} = \frac{\sin A}{\sin a}$, and then obtain the corresponding value of C by (1) or (2) of Napier's Analogies, and the value of c by (3) or (4).

In order that the problem may be possible, it is necessary that the $\sin B$ or $\sin b$ be comprised between zero and 1. When this condition is satisfied, then B or b will have two values, supplementary to each other. But it is necessary that the corresponding values of $\tan\frac{1}{2}C$ and $\tan\frac{1}{2}c$ be positive, which requires that $A-B$ and $a-b$ have the same sign. If this condition be not satisfied for either of the two values of B or b, the problem admits of no solution. But, if it is satisfied, a solution will necessarily follow, $A-B$ and $a-b$ being of the same sign, C and c will be comprised between zero and 180°, by means of two formulæ of Napier. These values of C and c will be the same as those which the other two formulæ of Napier will give.

Example. Given $a = 50^\circ\,45'\,20''$,
$b = 69^\circ\,12'\,40''$,
$A = 44^\circ\,22'\,10''$; to find B, C, and c.

Solution:

$$(1)\quad \sin B = \frac{\sin A \sin b}{\sin a}.$$

$$\begin{aligned}
\log \sin 69^\circ\,12'\,40'' &= \bar{1}.970763\\
\log \sin 44^\circ\,22'\,10'' &= \bar{1}.844652\\
\operatorname{colog} \sin 50^\circ\,45'\,20'' &= 0.111004\\
\log \sin B &= \bar{1}.926419
\end{aligned}$$

$$\therefore\ B = 57^\circ\,34'\,51''.4,$$

$$\text{or}\quad 122^\circ\,25'\,\ 8''.6.$$

There are then two solutions.

$$(2)\quad \cot \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(b+a)}{\cos \frac{1}{2}(b-a)} \tan \tfrac{1}{2}(A+B). \qquad (1) \text{ of } [61].$$

$$\begin{aligned}
\tfrac{1}{2}(B+A) &= 50^\circ\,58'\,30''.7 \qquad \log \cos\ 59^\circ\,59' &&= \bar{1}.699189\\
\tfrac{1}{2}(B-A) &= \ 6^\circ\,36'\,20''.7 \qquad \log \tan\ 50^\circ\,58'\,30''.7 &&= 0.091246\\
\tfrac{1}{2}(b+a) &= 59^\circ\,59' \qquad \operatorname{colog} \cos\ \ 9^\circ\,13'\,40'' &&= 0.005657\\
\tfrac{1}{2}(b-a) &= \ 9^\circ\,13'\,40'' \qquad \log \cot \tfrac{1}{2} C &&= \bar{1}.796082\\
&\qquad\qquad\qquad\qquad \tfrac{1}{2} C = \ 57^\circ\,58'\,55''.3.
\end{aligned}$$

When $B = \ 57^\circ\,34'\,51''.4$, then $C = 115^\circ\,57'\,50''.6$.

But when $B = 122^\circ\,25'\,\ 8''.6$,

then

$$\begin{aligned}
\tfrac{1}{2}(B+A) &= \ 83^\circ\,23'\,39''.3,\\
\tfrac{1}{2}(B-A) &= \ 39^\circ\ \ 1'\,29''.3,\\
\tfrac{1}{2}(b+a) &= \ 59^\circ\,59',\\
\tfrac{1}{2}(b-a) &= \ \ 9^\circ\,13'\,40''.
\end{aligned}$$

By (1) of [61],

$$\begin{aligned}
\log \cos\ 59^\circ\,59' &= \bar{1}.699189\\
\log \tan\ 83^\circ\,23'\,39''.3 &= 0.936270\\
\operatorname{colog} \cos\ \ 9^\circ\,13'\,40'' &= 0.005657\\
\log \cot \tfrac{1}{2} C &= 0.641116
\end{aligned}$$

$$\therefore\ \tfrac{1}{2} C = 12^\circ\,52'\,15''.8.$$

For second value of B, then $C = 25^\circ\,44'\,31''.6$.

(3) The values of c are found by (3) of [61].

$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(B+A)}{\cos \frac{1}{2}(B-A)} \tan \tfrac{1}{2}(b+a).$$

$$\begin{aligned}
\log \cos 83^\circ\ 23'\ 39''.3 &= \bar{1}.060837 \\
\log \tan 59^\circ\ 59' \qquad\ &= 0.238269 \\
\operatorname{colog} \cos 39^\circ\ \ 1'\ 29''.3 &= 0.109650 \\
\log \tan \tfrac{1}{2}c &= \bar{1}.408756 \\
\tfrac{1}{2}c &= 14^\circ\ 22'\ 32''.6. \\
\therefore\ \ c &= 28^\circ\ 45'\ \ 5''.2.
\end{aligned}$$

The other value of c is found by taking the other values of $\frac{1}{2}(B+A)$ and $\frac{1}{2}(B-A)$.

108. Case VI. *Given two angles and the side opposite one of them; to find the other parts.*

This case gives rise to the same ambiguities that are found in Case V.

To solve the problem, the formulæ to be used are

(1) $\sin b = \dfrac{\sin a \sin B}{\sin A}$;

(2) Napier's Analogies, [61].

109. *The two cases that admit of two solutions.*

Case V. and Case VI. are the only two cases of spherical triangles that admit of two solutions, and yet they do not *always* admit of such ambiguity. The formulæ which relate to these two cases make known the number of solutions, and determine without ambiguity the elements of each of them.

In order that the problem may be possible, it is necessary and sufficient that $\tan A$ and $\cos a$, $\cos A$ and $\tan a$, have the same sign; that is to say, that A and a be both less than 90°, or both greater than 90°. There is then but one solution.

Passing to the general case, it is necessary, in order that the problem be possible, that $\dfrac{\sin b \sin A}{\sin a}$ be less than 1; if this condition be satisfied, there are then two values of B that satisfy the equation $\sin B = \dfrac{\sin b \sin A}{\sin a}$, one of which is B and the other $180^\circ - B$.

Now, $A-B$ and $a-b$ must have the same sign, in order that B and $180°-B$ may satisfy the problem.

In the cases in which two solutions are indicated, no solution is possible if $\sin a$ be less than $\sin b \sin A$.

If a lies between b and $180-b$, there will be one solution.

If a does not lie between b and $180°-b$, either there are two solutions or no solution. The cases in which $a=b$, or $a=180°-b$, are not included in the last supposition.

110. EXAMPLES.

SET VIII.

1. Given $a = 43° 27' 36''$,
$b = 82° 58' 17''$,
$A = 29° 32' 29''$; to find B, C, and c.

In this problem, $A<90°$, and $b<90°$; therefore $a<b$ will give two solutions.

Take $$\sin B = \frac{\sin b \sin A}{\sin a}.$$

2. Given $a = 74° 23'$,
$b = 35° 46' 14''$,
$c = 100° 39''$; to find A, B, and C.

3. Given $A = 48° 30'$,
$B = 125° 20'$,
$C = 62° 54'$;
to find a, b, and c.

4. Given $a = 70° 14' 20''$,
$b = 49° 24' 10''$,
$c = 38° 46' 10''$;
to find A, B, and C.

5. Given $A = 129° 5' 28''$,
$B = 142° 12' 42''$,
$C = 105° 8' 10''$;
to find a, b, and c.

6. Given $a = 68° 46' 2''$,
$b = 37° 10'$,
$C = 39° 23' 23''$;
to find A, B, and c.

7. Given $A = 34° 15' 3''$,
$B = 42° 15' 13''$,
$c = 76° 35' 36''$;
to find a, b, and C.

8. Given $a = 97° 35'$,
$b = 27° 8' 22''$,
$A = 40° 51' 18''$;
to find B, C, and c.

9. Given $A = 50° 12'$,
$B = 58° 8'$,
$a = 62° 42'$;
to find b, c, and C.

How many solutions has (9)?

10. Given $a = 150° 17' 23''$,
$b = 43° 12'$,
$c = 82° 50' 12''$;
to find A, B, and C.

11. Given $A = 50°$,
$a = 40°$,
$b = 60°$;
to find B, C, and c.

12. Given $A = 135° 5' 28''.8$,
$C = 50° 30' 8''.4$,
$b = 69° 34' 55''.9$;
to find a, c, and B.

13. Given $A = 30° 28' 11''$,
$B = 130° 3' 11''$,
$c = 40°$;
to find a, b, and C.

14. Given $a = 68° 46' 2''$,
$b = 43° 37' 38''$,
$c = 37° 10'$;
to find A, B, and C.

15. Given $A = 31° 34' 26''$,
$B = 30° 28' 12''$,
$c = 70° 2' 3''$;
to find a, b, and C.

16. Given $a = 63° 50'$,
$b = 80° 19'$,
$A = 51° 30'$;
to find B, C, and c.

17. Given $A = 32° 26' 6''.66$,
$B = 130° 5' 22''$,
$a = 44° 13' 42''$;
to find b, c, and C.

18. Given $A = 120° 43' 37''$,
$B = 109° 55' 42''$,
$C = 116° 38' 33''$;
to find a, b, and c.

CHAPTER X.

AREA OF SPHERICAL TRIANGLES.

111. *Given the angles of a spherical triangle; to find its area.*

Let *ABC* (Fig. 31) be a spherical triangle traced upon the surface of a sphere, and *CAE*, *CBE*, *AED*, *ABD*, and *ACD* be semicircumferences. Then the triangle *ABC* will be a part of each of three lunes, *CAEB*, *CAB–BED*, and *ABDC*.

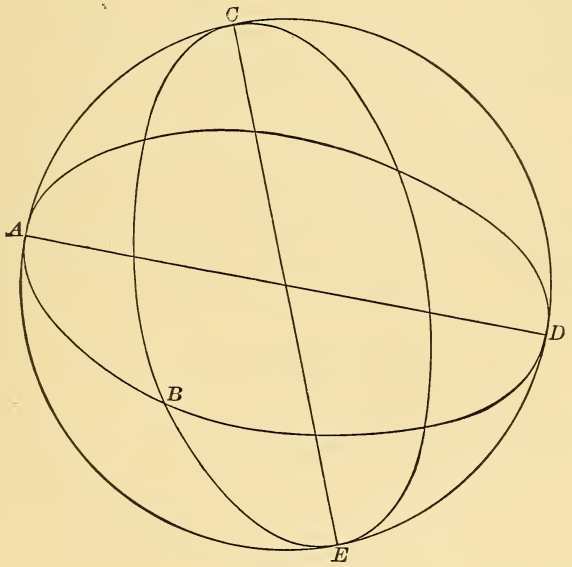


FIG. 31.

The surface of the hemisphere whose base is *ACDE*, is equal to the surface of the three lunes *less twice the triangle ABC*. Let r = the radius of the sphere.

(1) The surface of the hemisphere $= 2\pi r^2$,

(2) The surface of the lune $CAEB = 2Cr^2$,

(3) The surface of the lune $CAB–BED = 2Br^2$,

(4) The surface of the lune $ABDC = 2Ar^2$.

Therefore $$2ABC = 2(A+B+C-\pi)r^2;$$

or $$ABC = (A+B+C-\pi)r^2.$$

Denote $A+B+C-\pi$, which is called the spherical excess by E, the value of ABC then becomes Er^2.

But $$Er^2 = \frac{E}{2\pi} \times 2\pi r^2,$$

$$\therefore\ ABC = \frac{E}{180^\circ} \times 2\pi r^2 = \frac{E}{90^\circ} \times \pi r^2. \qquad [62]$$

112. *Given the three sides of a spherical triangle; to find its area.*

An elegant formula due to Simon l'Huillier, of Geneva, furnishes a direct method for the solution of this problem.

Since $$E = A+B+C-\pi,$$

then $$\tan\tfrac{1}{4}E = \frac{\sin\frac{1}{4}(A+B+C-\pi)}{\cos\frac{1}{4}(A+B+C-\pi)}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(C-\pi)}{\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(C-\pi)}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}C};$$

multiplying both terms of this fraction by $\cos\frac{1}{2}c$, it becomes

$$\frac{\sin\frac{1}{2}(A+B)\cos\frac{1}{2}c - \cos\frac{1}{2}c\cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B)\cos\frac{1}{2}c + \cos\frac{1}{2}c\sin\frac{1}{2}C},$$

which by (1) and (3) of [60],

$$= \frac{[\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c]\cos\frac{1}{2}C}{[\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c]\sin\frac{1}{2}C},$$

and replacing $\cos\frac{1}{2}C$ and $\sin\frac{1}{2}C$ by their values, [55] and [54], we have this last

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}},$$

which, by Pl. Tr.,

$$= \frac{\sin \frac{1}{4}(a-b+c)\sin \frac{1}{4}(b+c-a)}{\cos \frac{1}{4}(a+b+c)\cos \frac{1}{4}(a+b-c)} \sqrt{\frac{\sin s \sin(s-c)}{\sin(s-a)\sin(s-b)}},$$

and since $s = \frac{1}{2}(a+b+c)$,

this last fraction becomes

$$= \sqrt{\frac{\sin^2 \frac{1}{2}(s-b)\sin^2 \frac{1}{2}(s-a)\sin s \sin(s-c)}{\cos^2 \frac{1}{2} s \cos^2 \frac{1}{2}(s-c)\sin(s-a)\sin(s-b)}};$$

and since
$$\sin s = 2\sin \tfrac{1}{2} s \cos \tfrac{1}{2} s,$$
$$\sin(s-c) = 2\sin \tfrac{1}{2}(s-c)\cos \tfrac{1}{2}(s-c),$$
$$\sin(s-b) = 2\sin \tfrac{1}{2}(s-b)\cos \tfrac{1}{2}(s-b),$$
and
$$\sin(s-a) = 2\sin \tfrac{1}{2}(s-a)\cos \tfrac{1}{2}(s-a),$$

the foregoing radical then

$$= \sqrt{\frac{\sin^2 \frac{1}{2}(s-b)\sin^2 \frac{1}{2}(s-a)\sin \frac{1}{2} s \cos \frac{1}{2} s \sin \frac{1}{2}(s-c)\cos \frac{1}{2}(s-c)}{\cos^2 \frac{1}{2} s \cos^2 \frac{1}{2}(s-c)\sin \frac{1}{2}(s-a)\cos \frac{1}{2}(s-a)\sin \frac{1}{2}(s-b)\cos \frac{1}{2}(s-b)}};$$

therefore

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a)\tan \tfrac{1}{2}(s-b)\tan \tfrac{1}{2}(s-c)}. \quad [63]$$

If E in [62] be replaced by its value in [63], then the area of a spherical triangle may be easily obtained in terms of its three sides.

113. *To find the area of a spherical polygon in terms of its angles.*

Draw arcs of great circles from one of the vertices to the others; the polygon will then be divided into triangles, whose areas may be computed separately. Their sum will be the area of the polygon.

If n be the number of sides of a polygon, T the sum of all its angles, and P its area, then

$$P = [T - (n-2)\pi] r^2. \quad [64]$$

114. *To find the angular radius of a small circle circumscribed about a spherical triangle.*

Let OA, OB, and OC, the arcs of great circles, be drawn from O, the pole of the small circle ABC, to the vertices of the spherical triangle ABC (Fig. 32). Three isosceles spherical triangles are formed, having the sides of the given triangle for their bases.

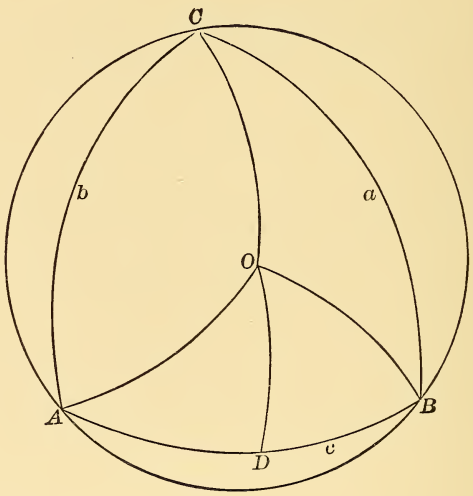


FIG. 32.

Let $\theta =$ each of the equal angles in the isosceles triangle OBA; θ', each equal angle in OAC; and θ'', each equal angle in OBC. Whether the pole of the circumscribed small circle be taken within or without the triangle, the formulæ will be the same.

Taking the pole within, then

$$\theta + \theta' = A; \quad \theta + \theta'' = B; \quad \text{and} \quad \theta' + \theta'' = C;$$

whence $$\theta + \theta' + \theta'' = \tfrac{1}{2}(A + B + C) = 90° + \tfrac{1}{2}E.$$

Therefore
$$\theta = 90° - (C - \tfrac{1}{2}E),$$
$$\theta' = 90° - (B - \tfrac{1}{2}E),$$
$$\theta'' = 90° - (A - \tfrac{1}{2}E).$$

Let the angular radius

$$OA = OB = OC = R.$$

Draw OD, an arc of a great circle, from O, perpendicular to the side c of the given triangle. The spherical right triangle ADO thus formed gives

$$\tan R \cos\theta = \tan\tfrac{1}{2}c.$$

Replacing θ by its value $90° - (C - \frac{1}{2}E)$, we obtain

$$\tan R = \frac{\tan\frac{1}{2}c}{\cos(90° - (C - \frac{1}{2}E))},$$

or

$$\tan R = \frac{\tan\frac{1}{2}c}{\sin(C - \frac{1}{2}E)}. \qquad [65]$$

115. *To find the angular radius of a small circle inscribed in a spherical triangle.*

Join the pole O, of the small circle EFD (Fig. 33), to the vertices, by the arcs of great circles OA, OB, and OC. Draw OF, OD, and OE, arcs of great circles, from O, perpendicular to the three sides of the triangle. Let $OF = r$, AOF being a right triangle, one of whose sides is r, another is $\frac{1}{2}(b + c - a)$ or $s - a$, and the angle OAF adjacent to $s - a$ equals $\frac{1}{2}A = \frac{1}{2}\theta$.

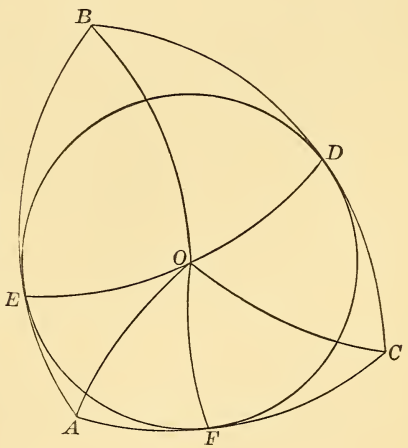


FIG. 33.

Then

$$\tan r = \sin(s - a)\tan\tfrac{1}{2}\theta,$$

or, replacing $\tan\frac{1}{2}\theta$ by its value in terms of the sides of ABC, we obtain

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \qquad [66]$$

116. *To find the angular radii of the small circles escribed upon the sides of a spherical triangle.*

An *escribed* circle is one which is tangent to one side of a triangle and the other two sides produced.

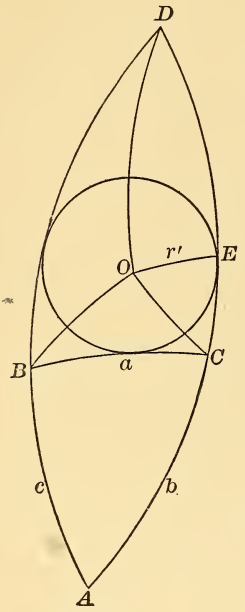


FIG. 34.

Prolong the sides b and c (Fig. 34) of the triangle ABC, until they meet at D, 180° from A. In this manner, another triangle BDC will be formed whose sides are

(1) a,

(2) $\pi - b$,

(3) $\pi - c$.

Let r', an arc of a great circle, be the radius required. From the right triangle DOE, we obtain, by letting A or its equal $D = \theta$,

$$\tan r' = \tan\frac{\theta}{2}\sin s.$$

Remembering that $s = \frac{1}{2}(a+b+c)$, and that $\tan\frac{1}{2}\theta$ is equal to

$$\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}},$$

then $\tan r'$

$$= \sin s\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} = \sqrt{\frac{\sin s\sin(s-b)\sin(s-c)}{\sin(s-a)}},$$

$\tan r''$

$$= \sin s\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}} = \sqrt{\frac{\sin s\sin(s-a)\sin(s-c)}{\sin(s-b)}},$$

$\tan r'''$

$$= \sin s\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}} = \sqrt{\frac{\sin s\sin(s-a)\sin(s-b)}{\sin(s-c)}}.$$ [67]

r'' and r''' are the angular radii of the two small circles tangent to the sides b and c and the prolongations of the others.

117. Examples.

SET IX.

1. Find the area of a spherical triangle, whose angles are 140°, 92°, and 68°, respectively, described on a sphere whose radius is 15 feet.

2. What is the area of a spherical triangle whose angles are 150°, 110°, and 60°, respectively, described on a sphere whose radius is 10 feet?

3. Given $a = 98°$, $b = 110°$, and $c = 115°$; to find the area of the triangle when traced on a sphere whose radius is 100 feet.

4. Each side of a spherical triangle is 10°. Required the spherical excess, the sphere's diameter being 10 feet.

5. Given $a = 88° 12' 20''$, $b = 124° 7' 17''$, and $C = 50° 2' 1''$; to find the spherical excess.

6. If the angles of a spherical triangle be together equal to four right angles, show that

$$\cos^2\tfrac{1}{2}a + \cos^2\tfrac{1}{2}b + \cos^2\tfrac{1}{2}c = 1.$$

7. A spherical polygon of five sides has $A = 75°$, $B = 80°$, $C = 115°$, $D = 120°$, and $E = 150°$. Find its area.

CHAPTER XI.

APPLICATIONS OF SPHERICAL TRIGONOMETRY.

118. The theory of Spherical Trigonometry has one of its most useful applications in the solution of astronomical problems. The theory owes its origin in no small degree to the inquiries that have grown out of the subject of Astronomy. Nearly all the lines that are considered as traced on the surface of the earth are, by astronomers, extended to the heavens, thus constituting a sphere called the *Celestial Sphere*, whose radius is "greater than any assignable quantity."

119. *Definitions of Terms.*

(1) The *Horizon* is a great circle traced on the celestial sphere, whose poles are called the *Zenith* and the *Nadir*. The plane of the horizon touches the surface of the earth at the point of observation, and is then called the *Sensible Horizon*. But when that plane passes through the centre of the earth parallel to the sensible horizon, it is called the *Rational Horizon*.

(2) The *Zenith* is a point in the celestial sphere vertically overhead.

(3) The *Nadir* is a point in the celestial sphere directly opposite to the zenith.

(4) *Vertical Circles* are great circles passing through the zenith and nadir, perpendicular to the horizon. Upon them the altitudes of celestial objects are measured.

(5) The *Meridian* is the great circle that passes through both the *North Pole* and the *South Pole*, and the *Zenith* and the *Nadir*.

(6) The *Prime Vertical* is the vertical circle cutting the meridian at right angles at the zenith, and, therefore, having an east and west direction on the celestial sphere.

120. The *Equinoctial* or *Celestial Equator* is a great circle traced on the celestial sphere by the plane of the earth's equator extended to the heavens.

(1) The *Axis* of the celestial sphere is the axis of the earth extended in both directions until it meets the sky.

(2) The *North Pole* and the *South Pole* are the ends of the axis of the celestial sphere.

(3) *Hour Circles*, sometimes called *Celestial Meridians*, are great circles of the celestial sphere passing through its two poles, and perpendicular to the equinoctial.

121. The *Ecliptic* is a great circle traced by the sun in its apparent annual motion about the earth. The *Ecliptic* and the *Equinoctial* intersect at an angle of nearly 23° 27′, at two points, one, called the *Vernal Equinox*, the other, the *Autumnal Equinox*. The date of the Vernal Equinox, i.e., when the sun crosses the equinoctial going north, is the 20th of March; that of the Autumnal Equinox, the 22d of September.

(1) *Circles of Celestial Latitude* are great circles that pass through the poles of the ecliptic, perpendicular to the plane of the ecliptic.

(2) The point on the ecliptic from which celestial longitude is estimated is the vernal equinox, towards the east.

122. The position of a star in the celestial sphere may be described in several ways by means of great circles and their poles taken as standards of reference. Whatever these standards may be, the quantities employed are called the *Spherical Co-ordinates* of the star.

The standards of reference and the spherical co-ordinates are as follows: —

(1) Horizon and Zenith . . . . . { Azimuth, and Altitude or Zenith Distance.

(2) Equinoctial and North Pole . . . { Right Ascension, and Declination; or, Declination, and Hour-Angle.

(3) Ecliptic and Pole . . . . . { Latitude, and Longitude.

Let *ENWS* be the plane of the horizon (Fig. 35); *Z*, the zenith; *O*, the point of observation; *WZE*, the prime vertical; *NZS*, the meridian; and *B*, any star.

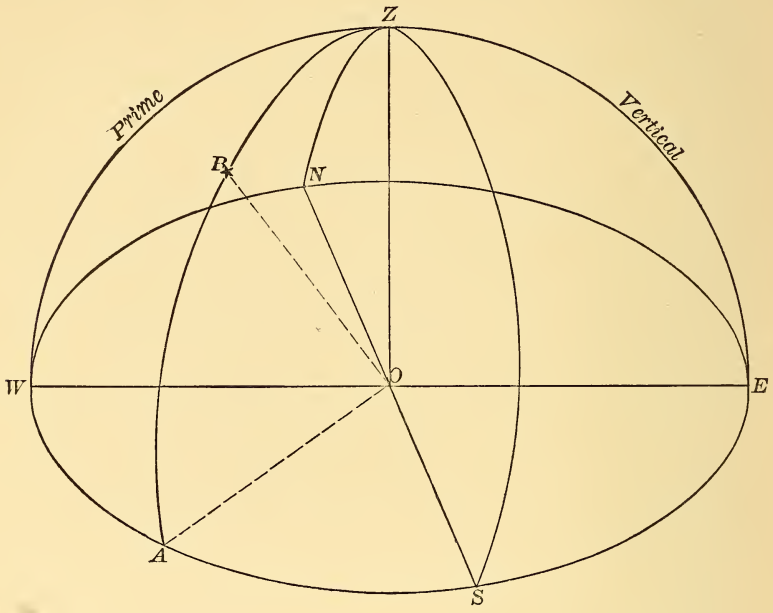


Fig. 35.

(1) The *Azimuth* of a star is its angular distance from either the north or the south point of the horizon to a vertical circle passing through the star.

The *Azimuth* of *B*, from the south point *S*, is the arc *AS*.

(2) The *Altitude* of a star is its angular elevation above the horizon, measured on a vertical circle passing through the star.

The *Altitude* of *B* is the arc *AB*. The *Zenith Distance* of *B* is *BZ*. The altitude + the zenith distance = 90°.

(3) The *Right Ascension* of a star is the arc of the equinoctial included between the vernal equinox and the foot of the hour-circle passing through the star.

Let *MWTE* (Fig. 36) be the equinoctial; *B*, the vernal equinox; *P* and *R*, the poles of the equinoctial; *POR*, the axis of the celestial sphere; *SWNE*, the horizon; and *A*, any star.

The *Right Ascension* of the star *A* is the angle *BPD*, or the arc *BD* measured from the *vernal equinox B*.

The *Hour-Angle* of the star *A* is the angle *DPM*, or the arc *MD*.

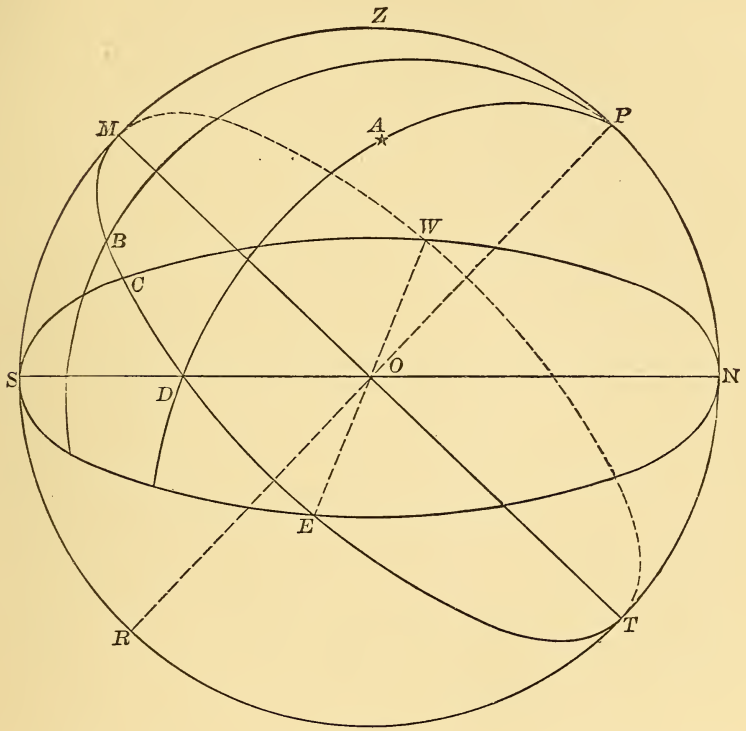


FIG. 36.

(4) The *Declination* of a star is its angular distance north or south of the *equinoctial*, and is measured by that arc of the *hour-circle* that extends from the equinoctial to the star.

The *North Polar Distance* is the arc of the *hour-circle* that extends from the star to the pole.

The *Declination* of the star *A* is the arc *AD*, and the *North Polar Distance* is the arc *AP*.

Let *EBFD* (Fig. 37) be the equinoctial; *ABCD*, the ecliptic; *P*, the pole of the equinoctial; *Z*, the pole of the ecliptic; *B*, the vernal equinox; *CBF*, the obliquity of the ecliptic; *S*, any star; *PSK*, the *hour-circle;* and *ZSM*, the *circle of latitude* through *S*.

(1) The *Latitude* of a star is its angular distance from the star to the ecliptic, measured on a *circle of latitude.*

The arc SM is the *latitude* of the star S.

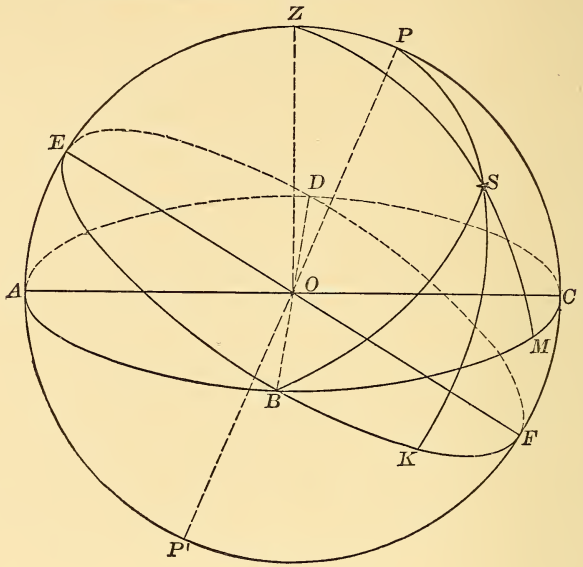


Fig. 37.

(2) The *Longitude* of a star measures the arc between the vernal equinox and the point on the ecliptic cut by the circle of latitude through the star.

The arc BM is the *longitude* of the star S.

123. *Given the sun's right ascension and declination ; to determine his longitude and the obliquity of the ecliptic.*

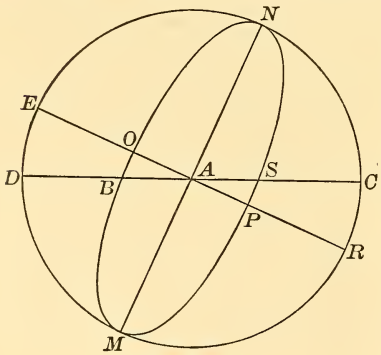


Fig. 38.

Let $EDRCN$ be the celestial meridian, passing through the points where the sun's declination is greatest; NM, the axis of the sphere; ER, the equator or equinoctial; DC, the ecliptic; NSM, the declination circle passing through the sun S. Then APS is a spherical triangle, right-angled at P. The right ascension is AP; the declination, SP;

and we are to find the longitude AS, and the obliquity SAP. *Solve by Napier's Rules for Circular Parts.*

124. *Given the sun's declination; to find the time of his rising at any place whose latitude is known.*

Let $NEMR$ (Fig. 39) be the meridian of the place; Z, the zenith; HO, the horizon; BC, the apparent path of the sun on the proposed day, cutting the horizon in S. EZ, then, will be the latitude, and EH or OR will be the complement of the latitude. EH or its equal measures the angle OAR. PS is the sun's declination, and AP, expressed in time, will be the time of sunrise. Degrees are changed into hours by dividing by 15. Therefore in the spherical right triangle APS, we have given PS, the declination, the angle SAP, to find AP, the time from 6 o'clock. Solve by Napier's Circular Parts.

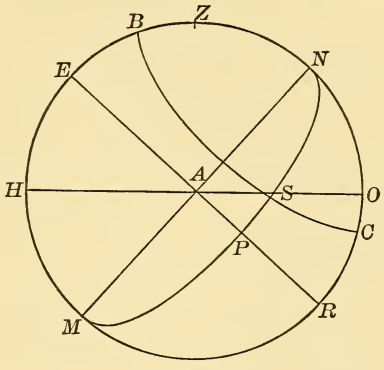


FIG. 39.

125. *Given the obliquity of the ecliptic and the declination of the sun; to find his longitude and right ascension.*

Let OB (Fig. 40) represent the ecliptic; OC, the equinoctial; and P, the pole of the equinoctial. Let A be the sun's position, and PAD the arc of a great circle passing through the sun and the pole. Then OAD is a spherical triangle, right-angled at D, and AD is the sun's declination, AO his longitude, and AOD the obliquity of the ecliptic. AD and the angle AOD are given, and it is required to determine the *longitude* and *right ascension.*

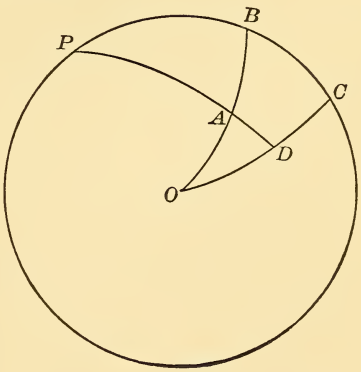


FIG. 40.

126. *Given the hour-angle, latitude, and declination of any star; to determine the altitude and azimuth.*

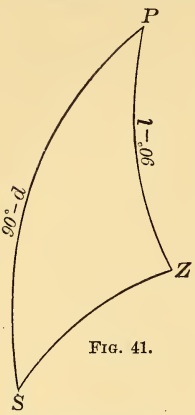


Fig. 41.

In Fig. 41, PS = the complement of the declination, PZ equals the complement of the latitude, and the angle SPZ, the hour-angle. The angle SZP is the supplement of the azimuth, and the arc SZ is the complement of the altitude. P is the pole; Z, the zenith; and S, the star. This problem gives $90° - d$, $90° - l$, and the hour-angle SPZ, from which to determine the altitude and the azimuth.

The angle SZP may be found by (1) and (2) of Napier's Analogies; then by the law of sines SZ can readily be obtained.

127. Examples.

SET X.

1. Given the sun's right ascension on a certain day, 53° 38′, and declination 19° 15′ 57″; to determine his longitude and the obliquity of the ecliptic.

2. On a certain day the sun's declination was observed to be 4° 13′ 31½″, the obliquity of the ecliptic being 23° 27′ 51″; required his right ascension.

3. Given the sun's declination 23° 28′; what is the time of sunrise at latitude 52° 13′ N.?

4. Required the time of sunrise at latitude 57° 2′ 54″, when the sun's declination is 23° 28′ N.

5. The obliquity of the ecliptic on a certain day was 23° 27′ 36″, and the declination of the sun 21° 52′ 56″. Find the sun's longitude and right ascension.

6. On a certain day the declination of the sun was 5° 58′ 8″, and the obliquity of the ecliptic 23° 27′ 38″. Find the sun's longitude and right ascension.

7. The latitude of a star is 40° 36′ 23″.9, the declination 23° 4′ 24″.3, and the hour-angle 46° 40° 4″.5. Find the star's altitude and azimuth.

8. The latitude of a star is 40° 36′ 23″.9, its altitude 47° 15′ 18″.3, and the azimuth 80° 23′ 4″.47. Find the star's declination and hour-angle.

CHAPTER XII.

MISCELLANEOUS EXAMPLES.

128. SET XI.

1. The sides of a triangle are 17, 21, and 28. Prove that the length of a line bisecting the longest side and drawn from the opposite angle is 13.

2. Show that the area of a quadrilateral, whose diagonals a and b intersect at an angle A, is $\frac{1}{2}ab\sin A$.

3. If $\theta+\theta'+\theta''=180°$, prove that

   (a) $\sin 2\theta+\sin 2\theta'+\sin 2\theta''=4\sin\theta\sin\theta'\sin\theta''$;

   (b) $\sin(\theta+\theta')\sin(\theta'+\theta'')=\sin\theta\sin\theta''$.

4. If $\theta+\theta'+\theta''=90°$, prove that

   (a) $\cos 2\theta+\cos 2\theta'+\cos 2\theta''=1+4\sin\theta\sin\theta'\sin\theta''$;

   (b) $\tan\frac{1}{2}\theta+\tan\frac{1}{2}\theta'+\tan\frac{1}{2}\theta''-\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\tan\frac{1}{2}\theta''$
   $=1-\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'-\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta''-\tan\frac{1}{2}\theta'\tan\frac{1}{2}\theta''$.

5. If $\tan\theta=\frac{1}{3}$, and $\tan\theta'=\frac{1}{7}$, show that $2\theta+\theta'=45°$.

6. If $\cot 2\theta=-\tan\theta'$, show that $\tan(\theta-\theta')=\cot\theta$.

7. If $\cos 3\theta+\cos 2\theta+\cos\theta=0$, show that $\theta=45°$, or $120°$, or $135°$, etc.

8. In a triangle ABC, right-angled at C, show that

$$\sin^2\tfrac{1}{2}A=\frac{c-b}{2c};\quad \cos^2\tfrac{1}{2}A=\frac{c+b}{2c};\quad \tan^2\tfrac{1}{2}A=\frac{c-b}{c+b}.$$

9. In a triangle ABC, right-angled at C, show that

$$\sin 2A=\frac{2ab}{a^2+b^2},\quad \text{and}\quad \cos 2A=\frac{a^2-b^2}{a^2+b^2}.$$

10. In a triangle ABC, right-angled at C, show that the

$$\text{area}=\tfrac{1}{4}c^2\sin 2A=\tfrac{1}{2}a^2\tan B=\tfrac{1}{2}b^2\tan A.$$

11. If $\cos B \sin C = \sin A$, show that the triangle will be isosceles.

12. The triangle ABC has its angles A, B, and C, in the proportion of the numbers 2, 3, and 4, respectively; show that

$$\cos \tfrac{1}{2} A = \frac{a+c}{2b}, \quad \text{and} \quad b^2 = \frac{a(a+c)^2}{2a+c}.$$

13. $\tan^{-1}\dfrac{1}{x-1} - \tan^{-1}\dfrac{1}{x+1} = \dfrac{\pi}{12}$; show that $x = \sqrt{3}+1$.

14. If $(s-a)(s-b) = ab$, show that the triangle is impossible.

15. The area of a triangle is 84 square inches, and two of its sides are 15 and 13 inches, respectively; find the third side.

16. The sides of a triangle are 3, 7, and 8, respectively. Compare the radii of the inscribed and circumscribed circles.

17. Two sides of a triangle are 8 and 10 inches, respectively, and the included angle 30°. Find the area.

18. A, B, C, D are four trees in a row, such that AB, BC, CD subtend equal angles at a point P. If $AB = 40$ feet, $BC = 20$ feet, and $CD = 60$ feet, show that $PA = 24\sqrt{5}$ feet, $PB = 8\sqrt{10}$ feet, $PC = 12\sqrt{5}$ feet, and $PD = 24\sqrt{10}$ feet.

19. A lighthouse 60 feet high is just seen from the deck of a ship 12 feet above the water; how far is the ship from the lighthouse?

20. If the length of an arc of 60° is 11 feet, show that the radius of the circle is 10 feet 6 inches.

21. The supplement of one angle of a triangle is double the complement of another, and triple that of the third; find the angles.

22. A ship sailing N. sees two lighthouses due E. After sailing an hour they are S.E. and S.S.E., the distance between them is 8 miles; find the rate of the ship.

23. Show that the number of acres in a field, whose sides are 400, 300, 300, and 300 yards, respectively, and one angle adjacent to the largest side is a right angle, is $\dfrac{125}{242}(5\sqrt{11}+24)$.

24. In Fig. 42, the line $ED = 200$ yards, $DC = 200$ yards, and $EF = 200$ yards. The angle $AFE = 83°$, $BDE = 156° 25'$, $BDC = 54° 30'$, $AED = 53° 30'$, $AEF = 54° 31'$, and $BCD = 88° 30'$. The length of AB is required.

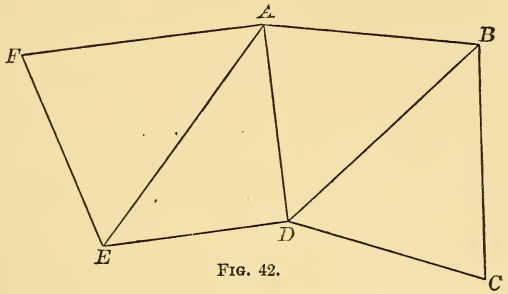


Fig. 42.

25. Given $AB = 800$ yards, $AC = 600$ yards, and $BC = 400$ yards. The angle $APC = 33° 45'$, and $BPC = 22° 30'$. Find the distances AP, CP, and BP. (Fig. 43.)

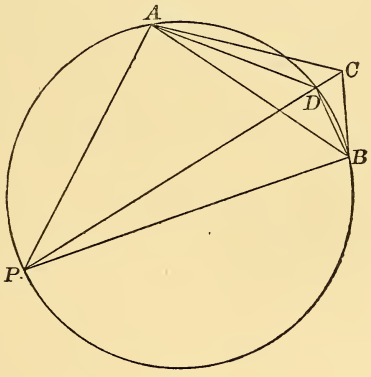


Fig. 43.

26. A balloon is observed from two stations 3000 feet apart. At the first station the horizontal angle of the balloon and the other station is $75° 15'$, and the elevation of the balloon is $18°$. The horizontal angle of the first station and the balloon, measured at the second station, is $64° 30'$. Find the height of the balloon.

27. In a spherical equilateral triangle, show that

$$2 \cos \frac{a}{2} \sin \frac{A}{2} = 1.$$

28. In a spherical triangle, if $b + c = \pi$, show that

$$\sin 2B + \sin 2C = 0.$$

29. ABC is a spherical right triangle, in which A is not the right angle. Show that if $A = a$, then c and b are quadrants.

30. In a spherical triangle, if $A = \frac{\pi}{2}$, $B = \frac{\pi}{5}$, and $C = \frac{\pi}{3}$, show that $a + b + c = \frac{\pi}{2}$.

31. A spherical square is divided into four equal right triangles by two diagonal arcs. Find the angle A of the square and one of its sides a.

32. If c_1 and c_2 be the two values of the third side of a spherical triangle, when A, a, and b, are given, and the triangle is ambiguous, show that

$$\tan \tfrac{1}{2} c_1 \tan \tfrac{1}{2} c_2 = \tan \tfrac{1}{2} (b - a) \tan \tfrac{1}{2} (b + a).$$

33. If the equal sides of a spherical isosceles triangle ABC be bisected by an arc DE, and BC be the base, show that

$$\sin \tfrac{1}{2} DE = \tfrac{1}{2} \sin \tfrac{1}{2} BC \sec \tfrac{1}{2} AC.$$

34. The sides of a spherical triangle are 105°, 90°, and 75°, respectively; find the sines of all the angles.

35. Find the area of a regular spherical polygon, whose angles are 50°, 95°, 130°, 140°, and 160°, respectively, on the surface of a sphere whose radius is 15 feet.

CHAPTER XIII.

TRIGONOMETRIC TABLES.

129. In order to use the trigonometric functions, it is necessary to compute the values of the functions of a given arc, and reciprocally be able to find the value of an arc when the value of one of its trigonometric functions is known. To reach this end, it is indispensable to have a table which will make known the values of the functions corresponding to the successive values of an arc comprised between 0° and $\frac{1}{2}\pi$, the intervals being sufficiently small.

Certain preliminary propositions must first be established, in order to show how such a table may be computed.

130. THEOREM I. *Every arc comprised between* 0° *and* $\frac{1}{2}\pi$ *is greater than its sine and less than its tangent.*

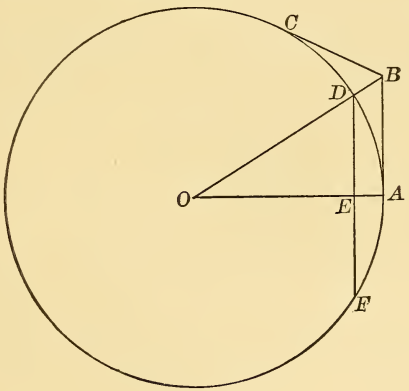


FIG. 44.

Let AD (Fig. 44) $= \theta$, be an arc comprised between 0° and $\frac{1}{2}\pi$; DE, the sine, and AB the tangent of θ. Prolong DE to F, and draw the tangent BC; we shall have the arc $DAF > DF$, and $ADC < AB + BC$.

The arc θ is one-half the arc DAF or the arc ADC; $\sin\theta$ is one-half DF, and $\tan\theta$ is equal to each one of the lines AB and BC; therefore $\theta > \sin\theta$, and $\theta < \tan\theta$.

131. Corollary. *If the arc θ decreases from $\frac{1}{2}\pi$ to 0, the ratio $\frac{\sin\theta}{\theta}$ approaches unity as a limit.*

Since $$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

we may write $$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta},$$

from which, dividing by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Hence it follows that the ratio $\frac{\theta}{\sin\theta}$ is comprised between *unity* and the *fraction* $\frac{1}{\cos\theta}$, whose limit is *unity* when $\theta = 0$;

$$\therefore \text{ limit of } \frac{\theta}{\sin\theta} = 1, \text{ or, limit of } \frac{\sin\theta}{\theta} = 1.$$

132. Theorem II. *The excess of an arc comprised between 0 and $\frac{1}{2}\pi$ over its sine is less than the one-fourth, and also less than the one-sixth, of the cube of that arc.*

(1) To show that $\theta - \sin\theta < \frac{\theta^3}{4}$, it will suffice to consider the inequality established in Theorem I.;

$$\tan\tfrac{1}{2}\theta > \tfrac{1}{2}\theta.$$

Multiplying by $$2\cos^2\tfrac{1}{2}\theta = 2(1 - \sin^2\tfrac{1}{2}\theta),$$

we obtain $$\sin\theta > \theta - \theta\sin^2\tfrac{1}{2}\theta,$$

whence $$\theta - \sin\theta < \theta\sin^2\tfrac{1}{2}\theta;$$

but $\sin\frac{1}{2}\theta$ is less than $\frac{1}{2}\theta$, and, consequently, $\sin^2\frac{1}{2}\theta$ is less than $\frac{1}{4}\theta^2$;

$$\therefore \theta - \sin\theta < \frac{\theta^3}{4}.$$

(2) To show that $$\theta - \sin\theta < \frac{\theta^3}{6}.$$

From Problem 11, Set IV., $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Replace θ successively by

$$\frac{\theta}{3}, \frac{\theta}{3^2}, \cdots, \frac{\theta}{3^n};$$

we obtain

$$3\sin\frac{\theta}{3} - \sin\theta = 4\sin^3\frac{\theta}{3},$$

$$3\sin\frac{\theta}{3^2} - \sin\frac{\theta}{3} = 4\sin^3\frac{\theta}{3^2},$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$3\sin\frac{\theta}{3^n} - \sin\frac{\theta}{3^{n-1}} = 4\sin^3\frac{\theta}{3^n};$$

multiplying these equations respectively by 1, 3, 3^2, $\cdots$, 3^{n-1}, and adding the results, the following equation is obtained,

$$3^n\sin\frac{\theta}{3^n} - \sin\theta = 4\left(\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + \cdots + 3^{n-1}\sin^3\frac{\theta}{3^n}\right);$$

or

$$\theta\,\frac{\sin\dfrac{\theta}{3^n}}{\left(\dfrac{\theta}{3^n}\right)} - \sin\theta = 4\left(\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + \cdots + 3^{n-1}\sin^3\frac{\theta}{3^n}\right).$$

If the integer n be indefinitely increased, the arc $\frac{\theta}{3^n}$ will tend towards 0, and the ratio

$$\frac{\sin\dfrac{\theta}{3^n}}{\dfrac{\theta}{3^n}}$$

towards unity; the first member of the preceding equality has then for its limit the difference $\theta - \sin\theta$, and consequently the second member tends also towards this limit.

But, as the sine is less than its arc, the limit of the second member is less than that towards which the geometrical progression

$$4\left(\frac{\theta^3}{3^3} + \frac{\theta^3}{3^5} + \frac{\theta^3}{3^7} + \cdots + \frac{\theta^3}{3^{2n+1}}\right) \quad \text{converges.}$$

The limit of this progression $= \frac{\theta^3}{6}$, consequently

$$\theta - \sin\theta < \frac{\theta^3}{6}.$$

133. The preceding theorems furnish two limits, that is θ and $\theta - \frac{\theta^3}{6}$, between which $\sin\theta$ is comprised. We may easily obtain two limits between which $\cos\theta$ is comprised.

We have $$\cos\theta = 1 - 2\sin^2\tfrac{1}{2}\theta,$$

and, as $\sin\theta$ is comprised between $\frac{1}{2}\theta$ and $\frac{1}{2}\theta - \frac{\theta^3}{48}$, we have at once

$$\cos\theta > 1 - \frac{\theta^2}{2},$$

since $$\cos\theta < 1 - 2\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)^2, \text{ or } < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - 2\left(\frac{\theta^3}{48}\right)^2,$$

and, for a stronger reason,

$$\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

Therefore $\cos\theta$ is comprised between

$$1 - \frac{\theta^2}{2} \text{ and } 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

134. *The construction of a table of sines and cosines.*

(1) Designate by θ the length of an arc of 10 seconds.

Since $$\pi = 3.1415926535897932\cdots$$

may be taken as the circumference of a circle whose radius equals unity, by dividing 10π by the number of seconds in a semicircumference, we obtain

$$\theta = \frac{10\pi}{648000} = 0.000048481368110\cdots.$$

By Art. 125, $$\sin\theta < \theta \text{ and } \sin\theta > \theta - \frac{\theta^3}{6};$$

also $$\frac{\theta^3}{6} < 0.000000000000021;$$

whence $$\sin 10'' < 0.000048481368110,$$

$$\sin 10'' > 0.000048481368089.$$

These two limits of $\sin 10''$ have the first twelve decimal figures common, and since they differ by less than half a unit in the thirteenth place, we may write

$$\sin 10'' = 0.0000484813681.$$

(2) To compute $\cos 10''$, we have, Art. 133,

$$\cos\theta > 1 - \frac{\theta^2}{2} \quad \text{and} \quad \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24};$$

and since θ is < 0.00005 or $< \frac{1}{2.10^4}$,

we have $$\frac{\theta^4}{24} < \frac{1}{384.10^{16}} < \frac{1}{3.10^{18}};$$

from which it follows that $1 - \frac{\theta^2}{2}$ is a value of $\cos\theta$ that differs from $1 - \frac{\theta^2}{2} + \frac{\theta^2}{24}$ by a quantity less than half a unit of the eighteenth decimal order.

Performing the operation for the first thirteen decimal places, we find

$$\cos 10'' = 0.9999999988248.$$

135. *Sines and cosines of arcs for every* $10''$, *up to* $45°$.

If, in the formulæ

$$\sin(\theta + \theta') + \sin(\theta - \theta') = 2\cos\theta' \sin\theta,$$
$$\cos(\theta + \theta') + \cos(\theta - \theta') = 2\cos\theta' \cos\theta,$$

we put $\theta = (m-1)\theta'$, the result will be

$$(1) \begin{cases} \sin m\theta' = 2\cos\theta' \sin(m-1)\theta' - \sin(m-2)\theta', \\ \cos m\theta' = 2\cos\theta' \cos(m-1)\theta' - \cos(m-2)\theta'. \end{cases}$$

If θ' be made equal to $10''$ and m to 2, then these formulæ will give the values of $\sin 20''$ and $\cos 20''$. Generally, if the sine and the cosine of two consecutive multiples of the arc $\theta' = 10''$ be known, formulæ (1) will make known the sine and the cosine of the following multiple arc. That is to say, we may suppose θ' to be

constantly equal to 10″, and m to be successively equal to 2, 3, 4, etc. We shall then obtain for the sines

$$\sin 20'' = 2\cos 10''\sin 10'' - \sin\ 0'' = .0000969627361$$
$$\sin 30'' = 2\cos 10''\sin 20'' - \sin 10'' =$$
$$\sin 40'' = 2\cos 10''\sin 30'' - \sin 20'' =$$
$$\text{Etc.,}$$

and for the cosines

$$\cos 20'' = 2\cos 10''\cos 10'' - \cos\ 0'' =$$
$$\cos 30'' = 2\cos 10''\cos 20'' - \cos 10'' =$$
$$\cos 40'' = 2\cos 10''\cos 30'' - \cos 20'' =$$
$$\text{Etc.}$$

The computation may be abridged in the following manner. Since the constant multiplier $2\cos 10''$ differs but little from 2,

then by placing $\quad 2\cos 10'' = 2 - k,$

we obtain $\quad k = 0.0000000023504.$

The sines may then be written

$$\sin 20'' = 2\sin 10'' - \sin\ 0'' - k\sin 10'',$$
$$\sin 30'' = 2\sin 20'' - \sin 10'' - k\sin 20'',$$
$$\sin 40'' = 2\sin 30'' - \sin 20'' - k\sin 30'', \text{ etc.,}$$

and the cosines

$$\cos 20'' = 2\cos 10'' - \cos\ 0'' - k\cos 10'',$$
$$\cos 30'' = 2\cos 20'' - \cos 10'' - k\cos 20'',$$
$$\cos 40'' = 2\cos 30'' - \cos 20'' - k\cos 30'', \text{ etc.}$$

This method of computing the sines and cosines of all arcs from 0″ to 30° inclusive, at intervals of 10″, is a very simple one.

136. The sines and cosines of angles or arcs above 30° up to and including 45°, are readily obtained by subtraction.

Remembering that $\sin 30° = \frac{1}{2}$, we have

$$\sin(30° + \theta) + \sin(30° - \theta) = \cos\theta,$$
$$\cos(30° - \theta) - \cos(30° + \theta) = \sin\theta,$$

whence $\quad \sin(30° + \theta) = \cos\theta - \sin(30° - \theta),$

$$\cos(30° + \theta) = \cos(30° - \theta) - \sin\theta.$$

If θ be made equal to $10''$, $20''$, $30''$, etc., successively, then

$$\sin(30°+10'') = \cos 10'' - \sin(30°-10''),$$
$$\sin(30°+20'') = \cos 20'' - \sin(30°-20''),$$
$$\sin(30°+30'') = \cos 30'' - \sin(30°-30''), \text{ etc.,}$$

and

$$\cos(30°+10'') = \cos(30°-10'') - \sin 10'',$$
$$\cos(30°+20'') = \cos(30°-20'') - \sin 20'',$$
$$\cos(30°+30'') = \cos(30°-30'') - \sin 30'', \text{ etc.}$$

This process may be continued up to and including 45°.

The tables for tangents, cotangents, secants, and cosecants can be constructed by means of the formulæ

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \sec\theta = \frac{1}{\cos\theta},$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

137. The sines and cosines of angles or arcs above 45° will be found by taking the cosines and sines of their complements below 45°, according to the formulæ of Art. 17.

138. The foregoing method is simple in principle, but laborious. A much more rapid and simple method is by infinite series.

TABLES OF LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

139. In numerical applications, computations are very often abbreviated by the use of logarithmic tables; hence there is much more need of knowing the logarithms of the sines, cosines, etc., of angles or arcs, than the *natural functions* whose development is shown by the preceding Articles.

If we take the logarithms of the natural sines and cosines of all the angles or arcs from $0''$ to 90° inclusive, another table, called the table of logarithmic sines and cosines, will be formed. This table once constructed, we may easily form a table of logarithms of tangents and cotangents by means of the formulæ

$$\log\tan\theta = \log\sin\theta - \log\cos\theta,$$
$$\log\cot\theta = \log\cos\theta - \log\sin\theta.$$

When desirable the logarithms of secants and cosecants are found by means of the formulæ

$$\log \operatorname{cosec} \theta = 1 - \log \cos \theta,$$

$$\log \operatorname{cosec} \theta = 1 - \log \sin \theta.$$

140. The logarithms of sines and cosines are never positive quantities; the logarithms of tangents of angles less than 45°, and of cotangents of angles greater than 45°, are negative; therefore, to avoid negative quantities in the tables, 10 is added to the logarithm of every trigonometric function, thus forming the logarithms of the tables. This increase of the real logarithms by 10 must always be taken into consideration in logarithmic computations.

141. *Definitions.*

(1) *Natural Numbers* are arithmetical numbers.

(2) *Tables of Natural Functions* are the natural numbers representing the values of sines, cosines, etc., when radius is taken equal to unity.

(3) The *Logarithm* of a natural number is the exponent of the power to which another number must be raised to produce the first number.

(4) The *Base* is the number whose power is to be obtained.

(5) The base of the Napierian system of logarithms is represented by the letter e, whose value $= 2.71828\cdots$.

(6) The base of the common system is 10.

(7) A logarithm consists of two parts — *characteristic* and *mantissa*. The characteristic is *integral*, and the mantissa *decimal*.

142. *Properties of Logarithms.*

(1) *The logarithm of a product equals the sum of the logarithms of its factors.*

(2) *The logarithm of a quotient equals the difference between the logarithm of the dividend and that of the divisor.*

(3) *The logarithm of any power of a number equals the logarithm of the number multiplied by the exponent of the power.*

(4) *The logarithm of any root of a number equals the logarithm of the number divided by the index of the root.*

143. The *Arithmetical Complement* of a logarithm is the remainder after subtracting the logarithm from zero.

The arithmetical complement, or co-log, as it is frequently called, is used when addition is substituted for subtraction, on the principle that *adding the co-logarithm of a number is the same precisely as subtracting the logarithm.*

144. The logarithmic and trigonometric tables that may be consulted present some variety in their mode of arrangement, and are usually accompanied with full explanation of their peculiarities and the methods of using the tables. It is not necessary to enter into any minute account of the way in which tables may be used with the greatest advantage. The student is referred to the explanations accompanying the tables to be used.

APPENDIX.

THE formulæ of the preceding pages are of great importance. Collected and numbered the same as in the text, they will be found convenient for reference and use.

PLANE TRIGONOMETRY.

Trigonometric ratios.

PAGE 6.

$$\left.\begin{array}{ll} (1) & \sin A = \frac{a}{c}, \\ (2) & \cos A = \frac{b}{c}, \\ (3) & \tan A = \frac{a}{b}, \\ (4) & \cot A = \frac{b}{a}, \\ (5) & \sec A = \frac{c}{b}, \\ (6) & \operatorname{cosec} A = \frac{c}{a}, \\ (7) & \operatorname{vers} A = 1 - \cos A, \\ (8) & \operatorname{covers} A = 1 - \sin A. \end{array}\right\} \quad [1]$$

$$\left.\begin{array}{ll} (1) & \tan A \times \cot A = 1, \\ (2) & \sin A \times \operatorname{cosec} A = 1, \\ (3) & \cos A \times \sec A = 1, \\ (4) & \tan A = \frac{\sin A}{\cos A}, \\ (5) & \cot A = \frac{\cos A}{\sin A}. \end{array}\right\} \quad [2]$$

PAGE 7.

$$\left.\begin{aligned}
&(1) && \sin A = \frac{a}{c} = \cos B,\\
&(2) && \cos A = \frac{b}{c} = \sin B,\\
&(3) && \tan A = \frac{a}{b} = \cot B,\\
&(4) && \cot A = \frac{b}{a} = \tan B,\\
&(5) && \sec A = \frac{c}{b} = \operatorname{cosec} B,\\
&(6) && \operatorname{cosec} A = \frac{c}{a} = \sec B.
\end{aligned}\right\} \quad [3]$$

PAGE 8.

$$\left.\begin{aligned}
&(1) && \sin^2 A + \cos^2 A = 1,\\
&(2) && 1 + \tan^2 A = \sec^2 A,\\
&(3) && 1 + \cot^2 A = \operatorname{cosec}^2 A,\\
&(4) && \cos A = \sqrt{1 - \sin^2 A},\\
&(5) && \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}},\\
&(6) && \cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A},\\
&(7) && \sec A = \frac{1}{\sqrt{1 - \sin^2 A}},\\
&(8) && \operatorname{cosec} A = \frac{1}{\sin A}.
\end{aligned}\right\} \quad [4]$$

Functions of the sum and the difference of two angles or arcs.

PAGE 24.

$$\sin(\theta + \theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta'. \quad [5]$$

$$\cos(\theta + \theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta'. \quad [6]$$

$$\sin(\theta - \theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta'. \quad [7]$$

$$\cos(\theta - \theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'. \quad [8]$$

$$\tan(\theta + \theta') = \frac{\tan\theta + \tan\theta'}{1 - \tan\theta\tan\theta'}. \quad [9]$$

PAGE 25.

$$\cot(\theta+\theta') = \frac{\cot\theta\cot\theta' - 1}{\cot\theta' + \cot\theta}. \qquad [10]$$

$$\tan(\theta-\theta') = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}. \qquad [11]$$

$$\cot(\theta-\theta') = \frac{1 + \cot\theta\cot\theta'}{\cot\theta' - \cot\theta}. \qquad [12]$$

Functions of the sum of three angles or arcs.

$$\left.\begin{aligned}\sin(\theta+\theta'+\theta'') = {}& \sin\theta\cos\theta'\cos\theta'' + \sin\theta'\cos\theta\cos\theta'' \\ & + \sin\theta''\cos\theta\cos\theta' - \sin\theta\sin\theta'\sin\theta''.\end{aligned}\right\} \quad [13]$$

$$\left.\begin{aligned}\cos(\theta+\theta'+\theta'') = {}& \cos\theta\cos\theta'\cos\theta'' - \cos\theta\sin\theta'\sin\theta'' \\ & - \cos\theta'\sin\theta\sin\theta'' - \cos\theta''\sin\theta\sin\theta''.\end{aligned}\right\} \quad [14]$$

PAGE 26.

$$\left.\begin{aligned}\sin(\theta+\theta') + \sin(\theta-\theta') &= 2\sin\theta\cos\theta', \\ \sin(\theta+\theta') - \sin(\theta-\theta') &= 2\cos\theta\sin\theta'. \\ \cos(\theta+\theta') + \cos(\theta-\theta') &= 2\cos\theta\cos\theta', \\ \cos(\theta+\theta') - \cos(\theta-\theta') &= -2\sin\theta\sin\theta'.\end{aligned}\right\} \quad [15]$$

Functions of the sum and the difference of the sines and cosines of two angles.

$$\left.\begin{aligned}&(1)\ \sin\phi + \sin\phi' = 2\sin\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi'), \\ &(2)\ \sin\phi - \sin\phi' = 2\sin\tfrac{1}{2}(\phi-\phi')\cos\tfrac{1}{2}(\phi+\phi'), \\ &(3)\ \cos\phi + \cos\phi' = 2\cos\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi'), \\ &(4)\ \cos\phi - \cos\phi' = 2\sin\tfrac{1}{2}(\phi+\phi')\sin\tfrac{1}{2}(\phi-\phi').\end{aligned}\right\} \quad [16]$$

$$\left.\begin{aligned}\cos\phi + \sin\phi' &= 2\sin\left(\tfrac{1}{4}\pi - \frac{\phi-\phi'}{2}\right)\cos\left(\tfrac{1}{4}\pi - \frac{\phi+\phi'}{2}\right), \\ \cos\phi - \sin\phi' &= 2\sin\left(\tfrac{1}{4}\pi - \frac{\phi+\phi'}{2}\right)\cos\left(\tfrac{1}{4}\pi - \frac{\phi-\phi'}{2}\right).\end{aligned}\right\} \quad [17]$$

Combinations formed from [16].

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$$\left.\begin{aligned}
&(1)\ \frac{\sin\phi+\sin\phi'}{\sin\phi-\sin\phi'}=\frac{\sin\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi')}{\sin\tfrac{1}{2}(\phi-\phi')\cos\tfrac{1}{2}(\phi+\phi')}=\frac{\tan\tfrac{1}{2}(\phi+\phi')}{\tan\tfrac{1}{2}(\phi-\phi')},\\
&(2)\ \frac{\sin\phi+\sin\phi'}{\cos\phi+\cos\phi'}=\frac{\sin\tfrac{1}{2}(\phi+\phi')}{\cos\tfrac{1}{2}(\phi+\phi')}=\tan\tfrac{1}{2}(\phi+\phi'),\\
&(3)\ \frac{\sin\phi+\sin\phi'}{\cos\phi-\cos\phi'}=\frac{\cos\tfrac{1}{2}(\phi-\phi')}{\sin\tfrac{1}{2}(\phi-\phi')}=\cot\tfrac{1}{2}(\phi-\phi'),\\
&(4)\ \frac{\sin\phi-\sin\phi'}{\cos\phi+\cos\phi'}=\frac{\sin\tfrac{1}{2}(\phi-\phi')}{\cos\tfrac{1}{2}(\phi-\phi')}=\tan\tfrac{1}{2}(\phi-\phi'),\\
&(5)\ \frac{\sin\phi'-\sin\phi'}{\cos\phi-\cos\phi}=\frac{\cos\tfrac{1}{2}(\phi+\phi')}{\sin\tfrac{1}{2}(\phi+\phi')}=\cot\tfrac{1}{2}(\phi+\phi'),\\
&(6)\ \frac{\cos\phi+\cos\phi'}{\cos\phi'-\cos\phi}=\frac{\cos\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi-\phi')}{\sin\tfrac{1}{2}(\phi+\phi')\sin\tfrac{1}{2}(\phi-\phi')}\\
&\qquad\qquad=\cot\tfrac{1}{2}(\phi+\phi')\cot\tfrac{1}{2}(\phi-\phi').
\end{aligned}\right\}\ [18]$$

Functions of double angles.

$$\sin 2\theta=2\sin\theta\cos\theta. \qquad [19]$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta. \qquad [20]$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}. \qquad [21]$$

$$\cot 2\theta=\frac{\cot^2\theta-1}{2\cot\theta}. \qquad [22]$$

Functions of half angles.

PAGE 28.

$$\sin\tfrac{1}{2}\theta=\pm\tfrac{1}{2}(\sqrt{1+\sin\theta}\mp\sqrt{1-\sin\theta}). \qquad [23]$$

$$\cos\tfrac{1}{2}\theta=\pm\tfrac{1}{2}(\sqrt{1+\sin\theta}\pm\sqrt{1-\sin\theta}). \qquad [24]$$

$$\cos\tfrac{1}{2}\theta=\pm\sqrt{\frac{1+\cos\theta}{2}}. \qquad [25]$$

$$\sin\tfrac{1}{2}\theta=\pm\sqrt{\frac{1-\cos\theta}{2}}. \qquad [26]$$

$$\tan\tfrac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}. \qquad [27]$$

$$\cot\tfrac{1}{2}\theta = \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}. \qquad [28]$$

$$\sin 2\theta = \pm 2\sin\theta\sqrt{1-\sin^2\theta} = \pm 2\cos\theta\sqrt{1-\cos^2\theta}. \qquad [29]$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1. \qquad [30]$$

Law of sines.

PAGE 31.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad [31]$$

PAGE 32.

$$\left.\begin{aligned} A + B + C &= 180°, \\ \frac{a}{\sin A} &= \frac{b}{\sin B}, \\ \frac{a}{\sin A} &= \frac{c}{\sin C}. \end{aligned}\right\} \qquad [32]$$

Law of cosines.

$$\left.\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C, \\ a^2 &= b^2 + c^2 - 2bc\cos A, \\ b^2 &= a^2 + c^2 - 2ac\cos B. \end{aligned}\right\} \qquad [33]$$

Side of a triangle in terms of the cosines of the adjacent angles and the other two sides.

PAGE 33.

$$\left.\begin{aligned} &(1)\ a = b\cos C + c\cos B, \\ &(2)\ b = c\cos A + a\cos C, \\ &(3)\ c = a\cos B + b\cos A. \end{aligned}\right\} \qquad [34]$$

Law of tangents.

PAGE 34.

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}. \qquad [35]$$

Functions of half angles in terms of the sides of a triangle.

PAGE 35.

$$\left.\begin{aligned} \sin\tfrac{1}{2}A &= \sqrt{\frac{(s-b)(s-c)}{bc}}, \\ \sin\tfrac{1}{2}B &= \sqrt{\frac{(s-a)(s-c)}{ac}}, \\ \sin\tfrac{1}{2}C &= \sqrt{\frac{(s-a)(s-b)}{ab}}. \end{aligned}\right\} \quad [36]$$

$$\left.\begin{aligned} \cos\tfrac{1}{2}A &= \sqrt{\frac{s(s-a)}{bc}}, \\ \cos\tfrac{1}{2}B &= \sqrt{\frac{s(s-b)}{ac}}, \\ \cos\tfrac{1}{2}C &= \sqrt{\frac{s(s-c)}{ab}}. \end{aligned}\right\} \quad [37]$$

PAGE 36.

$$\left.\begin{aligned} \tan\tfrac{1}{2}A &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \\ \tan\tfrac{1}{2}B &= \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \\ \tan\tfrac{1}{2}C &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \end{aligned}\right\} \quad [38]$$

Area of triangles.

$$K = \tfrac{1}{2}bc\sin A. \quad [39]$$

PAGE 37.

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \quad [40]$$

$$\left.\begin{aligned} &(1)\ \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = \frac{(s-a)(s-b)(s-c)}{abc} = \frac{K^2}{sabc}, \\ &(2)\ \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \frac{s\sqrt{s(s-a)(s-b)(s-c)}}{abc} = \frac{Ks}{abc}, \\ &(3)\ \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C \quad \cdots\cdots\cdots = \frac{K}{s^2}. \end{aligned}\right\} \quad [41]$$

Radii of circumscribed and inscribed circles.

PAGE 38.

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \qquad [42]$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad [43]$$

Radii of escribed circles.

PAGE 39.

$$\left.\begin{aligned} r' &= \frac{K}{s-a} = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \\ r'' &= \frac{K}{s-c} = \sqrt{\frac{s(s-a)(s-b)}{s-c}}, \\ r''' &= \frac{K}{s-b} = \sqrt{\frac{s(s-a)(s-c)}{s-b}}. \end{aligned}\right\} \qquad [44]$$

$$\left.\begin{aligned} r' &= s\tan\tfrac{1}{2}A, \\ r'' &= s\tan\tfrac{1}{2}C, \\ r''' &= s\tan\tfrac{1}{2}B. \end{aligned}\right\} \qquad [45]$$

PAGE 40.

$$\left.\begin{aligned} \frac{1}{r} &= \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}, \\ K &= \sqrt{rr'r''r'''}, \\ 4R &= r' + r'' + r''' - r. \end{aligned}\right\} \qquad [46]$$

SPHERICAL TRIGONOMETRY.

Fundamental formulæ of spherical triangles.

PAGE 51.

$$\left.\begin{aligned} \cos a &= \cos b\cos c + \sin b\sin c\cos A, \\ \cos b &= \cos a\cos c + \sin a\sin c\cos B, \\ \cos c &= \cos a\cos b + \sin a\sin b\cos C. \end{aligned}\right\} \qquad [47]$$

PAGE 52.

$$\left.\begin{aligned}\cos A &= -\cos B\cos C + \sin B\sin C\cos a,\\ \cos B &= -\cos A\cos C + \sin A\sin C\cos b,\\ \cos C &= -\cos A\cos B + \sin A\sin B\cos c.\end{aligned}\right\} \quad [48]$$

Law of sines.

PAGE 53.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \quad [49]$$

Three sides and two angles of a spherical triangle.

PAGE 54.

$$\left.\begin{aligned}\cos a\sin b - \sin a\cos b\cos C &= \sin c\ \cos A,\\ \cos b\ \sin a - \sin b\cos a\cos C &= \sin c\ \cos B,\\ \cos b\ \sin c - \sin b\cos c\cos A &= \sin a\cos B,\\ \cos c\ \sin b - \sin c\cos b\cos A &= \sin a\cos C,\\ \cos c\ \sin a - \sin c\cos a\cos B &= \sin b\ \cos C,\\ \cos a\sin c - \sin a\cos c\cos B &= \sin b\ \cos A.\end{aligned}\right\} \quad [50]$$

Two sides and three angles of a spherical triangle.

$$\left.\begin{aligned}\cos a\sin B - \cos b\cos C\sin A &= \cos A\sin C,\\ \cos b\sin A - \cos a\cos C\sin B &= \cos B\sin C,\\ \cos b\sin C - \cos c\cos A\sin B &= \cos B\sin A,\\ \cos c\ \sin B - \cos b\cos A\sin C &= \cos C\sin A,\\ \cos c\ \sin A - \cos a\cos B\sin C &= \cos C\sin B,\\ \cos a\sin C - \cos c\cos B\sin A &= \cos A\sin B.\end{aligned}\right\} \quad [51]$$

Two sides and two angles of a spherical triangle.

$$\left.\begin{aligned}\cot a\sin b - \cot A\sin C &= \cos b\cos C,\\ \cot b\ \sin a - \cot B\sin C &= \cos a\cos C,\\ \cot b\ \sin c - \cot B\sin A &= \cos c\ \cos A,\\ \cot c\ \sin b - \cot C\sin A &= \cos b\cos A,\\ \cot c\ \sin a - \cot C\sin B &= \cos a\cos B,\\ \cot a\sin c - \cot A\sin B &= \cos c\ \cos B.\end{aligned}\right\} \quad [52]$$

Formulæ for solving spherical right triangles.

PAGE 55.

$$\left.\begin{aligned}
&(1)\ \cos a = \cos b \cos c,\\
&(2)\ \sin b = \sin a \sin B,\\
&(3)\ \sin c = \sin a \sin C,\\
&(4)\ \tan b = \tan a \cos C,\\
&(5)\ \tan b = \sin c \tan B,\\
&(6)\ \tan c = \tan a \cos B,\\
&(7)\ \tan c = \sin b \tan C,\\
&(8)\ \cos a = \cot B \cot C,\\
&(9)\ \cos B = \cos b \sin C,\\
&(10)\ \cos C = \cos c \sin B.
\end{aligned}\right\} \quad [53]$$

Functions of half angles in terms of the sides of a spherical triangle.

PAGE 56.

$$\left.\begin{aligned}
\sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}},\\
\sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}},\\
\sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}.
\end{aligned}\right\} \quad [54]$$

$$\left.\begin{aligned}
\cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}},\\
\cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}},\\
\cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.
\end{aligned}\right\} \quad [55]$$

$$\left.\begin{aligned}
\tan\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}},\\
\tan\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}},\\
\tan\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}.
\end{aligned}\right\} \quad [56]$$

Functions of half sides in terms of the angles of a spherical triangle.

PAGE 58.

$$\left.\begin{aligned}
\sin\tfrac{1}{2}a &= \sqrt{-\frac{\cos S\cos(S-A)}{\sin B\sin C}},\\
\sin\tfrac{1}{2}b &= \sqrt{-\frac{\cos S\cos(S-B)}{\sin A\sin C}},\\
\sin\tfrac{1}{2}c &= \sqrt{-\frac{\cos S\cos(S-C)}{\sin A\sin B}}.
\end{aligned}\right\} \quad [57]$$

$$\left.\begin{aligned}
\cos\tfrac{1}{2}a &= \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}},\\
\cos\tfrac{1}{2}b &= \sqrt{\frac{\cos(S-A)\cos(S-C)}{\sin A\sin C}},\\
\cos\tfrac{1}{2}c &= \sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}}.
\end{aligned}\right\} \quad [58]$$

$$\left.\begin{aligned}
\tan\tfrac{1}{2}a &= \sqrt{-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}},\\
\tan\tfrac{1}{2}b &= \sqrt{-\frac{\cos S\cos(S-B)}{\cos(S-A)\cos(S-C)}},\\
\tan\tfrac{1}{2}c &= \sqrt{-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}.
\end{aligned}\right\} \quad [59]$$

Delambre's Formulæ.

PAGE 60.

$$\left.\begin{aligned}
&(1) \quad \frac{\sin\tfrac{1}{2}(A+B)}{\cos\tfrac{1}{2}C} = \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}c},\\
&(2) \quad \frac{\sin\tfrac{1}{2}(A-B)}{\cos\tfrac{1}{2}C} = \frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}c},\\
&(3) \quad \frac{\cos\tfrac{1}{2}(A+B)}{\sin\tfrac{1}{2}C} = \frac{\cos\tfrac{1}{2}(a+b)}{\cos\tfrac{1}{2}c},\\
&(4) \quad \frac{\cos\tfrac{1}{2}(A-B)}{\sin\tfrac{1}{2}C} = \frac{\sin\tfrac{1}{2}(a+b)}{\sin\tfrac{1}{2}c}.
\end{aligned}\right\} \quad [60]$$

Napier's Analogies.

$$\left.\begin{aligned}
&(1)\ \tan\tfrac{1}{2}(A+B)=\cot\tfrac{1}{2}C\,\frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)},\\
&(2)\ \tan\tfrac{1}{2}(A-B)=\cot\tfrac{1}{2}C\,\frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)},\\
&(3)\ \tan\tfrac{1}{2}(a+b)\ =\tan\tfrac{1}{2}c\,\frac{\cos\tfrac{1}{2}(A-B)}{\cos\tfrac{1}{2}(A+B)},\\
&(4)\ \tan\tfrac{1}{2}(a-b)\ =\tan\tfrac{1}{2}c\,\frac{\sin\tfrac{1}{2}(A-B)}{\sin\tfrac{1}{2}(A+B)}.
\end{aligned}\right\}\quad[61]$$

Area of spherical triangles.

PAGE 76.

$$ABC=\frac{E}{90^\circ}\times\pi r^2. \qquad [62]$$

l'Huillier's Formula.

PAGE 77.

$$\tan\tfrac{1}{4}E=\sqrt{\tan\tfrac{1}{2}s\,\tan\tfrac{1}{2}(s-a)\,\tan\tfrac{1}{2}(s-b)\,\tan\tfrac{1}{2}(s-c)}. \qquad [63]$$

Area of spherical polygons.

$$P=[T-(n-2)\pi]r^2. \qquad [64]$$

Angular radii of small circles circumscribed about, inscribed in, and escribed upon, the sides of spherical triangles.

PAGE 79.

$$\tan R=\frac{\tan\tfrac{1}{2}c}{\sin(C-\tfrac{1}{2}E)}. \qquad [65]$$

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \qquad [66]$$

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$$\left.\begin{aligned}
&\tan r' \\
&= \sin s \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}} = \sqrt{\frac{\sin s \sin(s-b)\sin(s-c)}{\sin(s-a)}}, \\
&\tan r'' \\
&= \sin s \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}} = \sqrt{\frac{\sin s \sin(s-a)\sin(s-c)}{\sin(s-b)}}, \\
&\tan r''' \\
&= \sin s \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}} = \sqrt{\frac{\sin s \sin(s-a)\sin(s-b)}{\sin(s-c)}}.
\end{aligned}\right\} [67]$$

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